

On the Gevrey strong hyperbolicity

Tatsuo Nishitani

Abstract

In this paper we are concerned with a homogeneous differential operator p of order m of which characteristic set of order m is assumed to be a smooth manifold. We define the Gevrey strong hyperbolicity index as the largest number s such that the Cauchy problem for $p + Q$ is well-posed in the Gevrey class of order s for any differential operator Q of order less than m . We study the case of the largest index and we discuss in which way the Gevrey strong hyperbolicity index relates with the geometry of bicharacteristics of p near the characteristic manifold.

1 Introduction

Let

$$P = D_0^m + \sum_{|\alpha| \leq m, \alpha_0 < m} a_\alpha(x) D^\alpha = p(x, D) + P_{m-1}(x, D) + \cdots$$

be a differential operator of order m defined near the origin of \mathbb{R}^{n+1} where $x = (x_0, \dots, x_n) = (x_0, x')$ and

$$D_j = -i\partial/\partial x_j, \quad D = (D_0, D'), \quad D' = (D_1, \dots, D_n).$$

Here $p(x, \xi)$ is the principal symbol of P ;

$$p(x, \xi) = \xi_0^m + \sum_{|\alpha|=m, \alpha_0 < m} a_\alpha(x) \xi^\alpha.$$

We assume that the coefficients $a_\alpha(x)$ are in the Gevrey class of order $s > 1$, sufficiently close to 1, which are constant outside $|x'| \leq R$. We say that $f(x) \in \gamma^{(s)}(\mathbb{R}^{n+1})$, the Gevrey class of order s , if for any compact set $K \subset \mathbb{R}^{n+1}$ there exist $C > 0, A > 0$ such that we have

$$|D^\alpha f(x)| \leq C A^{|\alpha|} |\alpha|!^s, \quad x \in K, \quad \forall \alpha \in \mathbb{N}^{n+1}.$$

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Definition 1.1. We say that the Cauchy problem for P is $\gamma^{(s)}$ well-posed at the origin if for any $\Phi = (u_0, u_1, \dots, u_{m-1}) \in (\gamma^{(s)}(\mathbb{R}^n))^m$ there exists a neighborhood U_Φ of the origin such that the Cauchy problem

$$\begin{cases} Pu = 0 & \text{in } U_\Phi, \\ D_0^j u(0, x') = u_j(x'), \quad j = 0, 1, \dots, m-1, & x' \in U_\Phi \cap \{x_0 = 0\} \end{cases}$$

has a unique solution $u(x) \in C^\infty(U_\Phi)$.

It is a fundamental fact that if $p(x, \xi)$ is strictly hyperbolic near the origin, that is $p(x, \xi_0, \xi') = 0$ has m real distinct roots for any x , near the origin and any $\xi' \neq 0$ then the Cauchy problem for $p + Q$ with any differential operator Q of order less than m is C^∞ well-posed near the origin. In particular, $\gamma^{(s)}$ well-posed for any $s > 1$. On the other hand the Lax-Mizohata theorem in the Gevrey classes asserts:

Proposition 1.1 ([16, Theorem 2.2]). *If the Cauchy problem for P is $\gamma^{(s)}$ ($s > 1$) well-posed at the origin then $p(0, \xi_0, \xi') = 0$ has only real roots ξ_0 for any $\xi' \in \mathbb{R}^n$.*

Taking this result into account we assume, throughout the paper, that $p(x, \xi_0, \xi') = 0$ has only real roots for any x near the origin and any $\xi' \in \mathbb{R}^n$.

Definition 1.2. We define $G(p)$ (the Gevrey strong hyperbolicity index) by

$$G(p) = \sup \left\{ 1 \leq s \mid \begin{array}{l} \text{Cauchy problem for } p + Q \text{ is } \gamma^{(s)} \text{ well-posed at the} \\ \text{origin for any differential operator } Q \text{ of order } < m \end{array} \right\}.$$

We first recall a basic result of Bronshtein [4].

Theorem 1.1 ([4, Theorem 1]). *Let p be a homogeneous differential operator of order m with real characteristic roots. Then for any differential operator Q of order less than m , the Cauchy problem for $p + Q$ is $\gamma^{(m/(m-1))}$ well-posed.*

This implies that for differential operators p of order m with real characteristic roots we have

$$G(p) \geq m/(m-1).$$

We also recall a result which bounds $G(p)$ from above. The following result is a special case of Ivrii [10, Theorem 1]. Recall that $(x, \xi) \in \mathbb{R}^{n+1} \times (\mathbb{R}^{n+1} \setminus \{0\})$ is called a characteristic of order r of p if

$$\partial_x^\alpha \partial_\xi^\beta p(x, \xi) = 0, \quad \forall |\alpha| + |\beta| < r.$$

Theorem 1.2 ([10, Theorem 1]). *Let p be a homogeneous differential operator of order m with real analytic coefficients and let $(0, \bar{\xi})$, $\bar{\xi} = (0, \dots, 0, 1) \in \mathbb{R}^{n+1}$ be a characteristic of order m . If the Cauchy problem for $P = p + P_{m-1} + \dots$ is $\gamma^{(\kappa)}$ well-posed at the origin we have*

$$\partial_{\bar{\xi}}^{\alpha} \partial_x^{\beta} P_{m-1}(0, \bar{\xi}) = 0$$

for any $|\alpha + \beta| \leq m - 2\kappa/(\kappa - 1)$.

Assume that p has a characteristic $(0, \bar{\xi})$ of order m and that the Cauchy problem for $p + P_{m-1} + \dots$ is $\gamma^{(\kappa)}$ well-posed for any P_{m-1} . Then from Theorem 1.2 it follows that $m - 2\kappa/(\kappa - 1) < 0$, that is $\kappa < m/(m - 2)$ which yields

$$G(p) \leq m/(m - 2).$$

Let ρ be a characteristic of order m . Then the localization $p_{\rho}(X)$ of p at ρ is defined by $p(\rho + \mu X) = \mu^m(p_{\rho}(X) + o(1))$ with $X = (x, \xi)$ as $\mu \rightarrow 0$ which is nothing but the first non-vanishing term of the Taylor expansion of p around ρ . Note that p_{ρ} is a hyperbolic polynomial in X in the direction $(0, \theta) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ where $\theta = (1, \dots, 0) \in \mathbb{R}^{n+1}$ (for example [7, Lemma 8.7.2]). The hyperbolic cone Γ_{ρ} of p_{ρ} is the connected component of $(0, \theta)$ in the set (for example [7, Lemma 8.7.3])

$$\Gamma_{\rho} = \{X \in \mathbb{R}^{2(n+1)} \mid p_{\rho}(X) \neq 0\}$$

and the propagation cone C_{ρ} of the localization p_{ρ} is given by

$$C_{\rho} = \{X \in \mathbb{R}^{2(n+1)} \mid (d\xi \wedge dx)(X, Y) \leq 0, \forall Y \in \Gamma_{\rho}\}.$$

Let $H_p = \sum_{j=0}^n (\partial p / \partial \xi_j) \partial / \partial x_j - (\partial p / \partial x_j) \partial / \partial \xi_j$ be the Hamilton vector field of p . The integral curves of H_p , along which $p = 0$, are called bicharacteristics of p . We note that C_{ρ} is the *minimal* cone including every bicharacteristic which has ρ as a limit point in the following sense:

Lemma 1.1 ([12, Lemma 1.1.1]). *Let $\rho \in \mathbb{R}^{n+1} \times (\mathbb{R}^{n+1} \setminus \{0\})$ be a multiple characteristic of p . Assume that there are simple characteristics ρ_j and non-zero real numbers γ_j with $\gamma_j p_{\rho_j}(0, \theta) > 0$ such that*

$$\rho_j \rightarrow \rho \quad \text{and} \quad \gamma_j H_p(\rho_j) \rightarrow X, \quad j \rightarrow \infty.$$

Then $X \in C_{\rho}$.

We now introduce assumptions of which motivation will be discussed in the next section. Denote by Σ the set of characteristics of order m of $p(x, \xi)$;

$$\Sigma = \{(x, \xi) \in \mathbb{R}^{n+1} \times (\mathbb{R}^{n+1} \setminus \{0\}) \mid \partial_x^\alpha \partial_\xi^\beta p(x, \xi) = 0, \forall |\alpha| + |\beta| < m\}$$

which is assumed to be a $\gamma^{(s)}$ manifold. Note that p_ρ is a function on $\mathbb{R}^{2(n+1)}/T_\rho \Sigma$ because $p_\rho(X + Y) = p_\rho(Y)$ for any $X \in T_\rho \Sigma$ and any $Y \in \mathbb{R}^{2(n+1)}$. We assume that

$$(1.1) \quad p_\rho \text{ is a strictly hyperbolic polynomial on } \mathbb{R}^{2(n+1)}/T_\rho \Sigma, \quad \rho \in \Sigma.$$

We also assume that the propagation cone C_ρ is transversal to the characteristic manifold Σ ;

$$(1.2) \quad C_\rho \cap T_\rho \Sigma = \{0\}, \quad \rho \in \Sigma.$$

Our aim in this paper is to prove

Theorem 1.3. *Assume (1.1) and (1.2). Then the Cauchy problem for $p+Q$ is $\gamma^{(s)}$ well-posed at the origin for any differential operator Q of order less than m and for any $1 < s < m/(m-2)$. In particular we have $G(p) = m/(m-2)$.*

Example 1.1. Let

$$q(\zeta) = \zeta_0^m + \sum_{|\alpha|=m, \alpha_0 \leq m-2} c_\alpha \zeta^\alpha, \quad \zeta = (\zeta_0, \zeta_1, \dots, \zeta_k)$$

be a strictly hyperbolic polynomial in the direction ζ_0 where $k \leq n$. Let $b_j(x, \xi')$, $j = 1, \dots, k$ be smooth functions in a conic neighborhood of $(0, \hat{\xi}')$ which are homogeneous of degree 1 in ξ' with linearly independent differentials at $(0, \hat{\xi}')$. We define

$$p(x, \xi) = q(b(x, \xi)), \quad b = (b_0, b_1, \dots, b_k)$$

where we set $b_0(x, \xi) = \xi_0$ for notational convenience. Then it is easy to see that $p(x, \xi)$ verifies the condition (1.1) near $\rho = (0, 0, \hat{\xi}')$ with $\Sigma = \{(x, \xi) \mid b_j(x, \xi) = 0, j = 0, \dots, k\}$ and $p_\rho(x, \xi) = q(db_\rho(x, \xi))$, that is

$$p_\rho(x, \xi) = q(\hat{b}(x, \xi)), \quad \hat{b} = (\hat{b}_0, \hat{b}_1, \dots, \hat{b}_k)$$

where $\hat{b}_j(x, \xi)$ is the linear part of $b_j(x, \xi)$ at ρ . Therefor $\Gamma_\rho = \{X \mid \hat{b}(X) \in \Gamma\}$ where Γ is the hyperbolic cone of q . If

$$(1.3) \quad (\{b_i, b_j\})_{0 \leq i, j \leq k} \text{ is non-singular at } \rho$$

then $p(x, \xi)$ verifies the condition (1.2) near ρ where $\{b_i, b_j\}$ denotes the Poisson bracket

$$\sum_{\mu=0}^n (\partial b_i / \partial \xi_\mu) (\partial b_j / \partial x_\mu) - (\partial b_i / \partial x_\mu) (\partial b_j / \partial \xi_\mu).$$

We check that (1.3) implies (1.2). Note that (1.2) is equivalent to $\Gamma_\rho \cap (T_\rho \Sigma)^\sigma \neq \emptyset$. Since $(T_\rho \Sigma)^\sigma$ is spanned by $H_{b_0}(\rho), H_{b_1}(\rho), \dots, H_{b_k}(\rho)$ it suffices to show that there are c_j such that $0 \neq X = \sum_{j=0}^k c_j H_{b_j}(\rho) \in \Gamma_\rho$. From

$$\hat{b}_j(X) = \sum_{i=0}^k c_i \{b_i, b_j\}(\rho), \quad j = 0, \dots, k$$

one can choose c_j so that $\hat{b}(X) = (1, 0, \dots, 0)$ by assumption (1.3) and hence the result.

Example 1.2. Consider

$$(1.4) \quad q(\zeta_0, \zeta_1, \zeta_2) = \prod_{j=1}^{\ell} (\zeta_0 - c_j(\zeta_1^2 + \zeta_2^2))$$

where c_j are real positive constants different from each other and $2\ell = m$. Take $b_1 = (x_0 - x_1)\xi_n$, $b_2 = \xi_1$ and consider

$$p(x, \xi) = \prod_{j=1}^{\ell} (\xi_0^2 - c_j((x_0 - x_1)^2 \xi_n^2 + \xi_1^2))$$

in a conic neighborhood of $\rho = (0, 0, \dots, 0, 1)$. The 3×3 anti-symmetric matrix $(\{b_i, b_j\})$ is obviously singular. If $\max\{c_j\} = c < 1$ then $C_\rho \cap T_\rho \Sigma = \{0\}$. To see this take any $X = (t, t, x_2, \dots, x_n, 0, 0, \xi_2, \dots, \xi_n) \in T_\rho \Sigma$. Assume $X \in C_\rho$ so that $(d\xi \wedge dx)(X, Y) \leq 0$ for any $Y = (y, \eta) \in \Gamma_\rho$, that is for any $(y, \eta) \in \mathbb{R}^{2(n+1)}$ with $\eta_0^2 > c((y_0 - y_1)^2 + \eta_1^2)$ and $\eta_0 > 0$. This implies that $x_2 = \dots = x_n = 0$, $\xi_2 = \dots = \xi_n = 0$ and $-t(\eta_0 + \eta_1) \leq 0$ for any $\eta_0 > \sqrt{c}|\eta_1|$. Since $c < 1$ this gives $t = 0$ so that $X = 0$.

On the other hand if $\max\{c_j\} = c \geq 1$ then $C_\rho \cap T_\rho \Sigma \neq \{0\}$. Indeed let $X = (1, 1, 0, \dots, 0, 0, \dots, 0) \in T_\rho \Sigma$. Noting that $\eta_0 > \sqrt{c}|\eta_1|$ if $Y = (y, \eta) \in \Gamma_\rho$ we see $(d\xi \wedge dx)(X, Y) = -\eta_0 - \eta_1 < 0$ for any $Y \in \Gamma_\rho$ which proves $X \in C_\rho$.

Example 1.3. Take q in (1.4) and choose $b_1 = x_0\xi_n$, $b_2 = \xi_1$ and consider

$$p(x, \xi) = \prod_{j=1}^{\ell} (\xi_0^2 - c_j(x_0^2\xi_n^2 + \xi_1^2))$$

near $\rho = (0, 0, \dots, 0, 1)$. As remarked in Example 1.2 the matrix $(\{b_i, b_j\})$ is singular. Suppose $X = (0, x_1, \dots, x_n, 0, 0, \xi_2, \dots, \xi_n) \in T_\rho\Sigma \cap C_\rho$. As in Example 1.2 we conclude $x_2 = \dots = x_n = 0$, $\xi_1 = \dots = \xi_n = 0$ and $-x_1\eta_1 \leq 0$ for any $\eta_0^2 > c(y_0^2 + \eta_1^2)$. This gives $x_1 = 0$ so that $X = 0$. Thus we conclude $C_\rho \cap T_\rho\Sigma = \{0\}$.

Example 1.4. We specialize Example 1.1 with

$$q(\zeta_0, \zeta_1) = \prod_{j=1}^m (\zeta_0 - \alpha_j \zeta_1), \quad q(\zeta_0, \zeta_1) = \prod_{j=1}^{\ell} (\zeta_0^2 - c_j \zeta_1^2)$$

where α_j are real constants different from each other such that $\sum_{j=1}^m \alpha_j = 0$ and c_j are positive constant different from each other and $m = 2\ell$. For these q choosing $b_1 = x_0\xi_1$ and $b_1 = x_0|\xi'|$ respectively we get

$$p(x, \xi) = \prod_{j=1}^m (\xi_0 - \alpha_j x_0 \xi_1), \quad p(x, \xi) = \prod_{j=1}^{\ell} (\xi_0^2 - c_j x_0^2 |\xi'|^2).$$

It is clear that $\{b_0, b_1\} = \xi_1 \neq 0$ and $\{b_0, b_1\} = |\xi'| \neq 0$ respectively and hence $C_\rho \cap T_\rho\Sigma = \{0\}$. We find these examples in [5] where they studied Levi type conditions for differential operators of order m with coefficients depending only on the time variable.

2 Motivation, the doubly characteristic case

In this section we provide the motivation to introduce $G(p)$ and assumptions (1.1), (1.2). Let $m = 2$ and we consider differential operators of second order

$$P(x, D) = p(x, D) + P_1(x, D) + P_0(x)$$

of principal symbol $p(x, \xi)$. Let ρ be a double characteristic of p and hence singular (stationary) point of H_p . We linearize the Hamilton equation $\dot{X} = H_p(X)$ at ρ , the linearized equation turns to be $\dot{Y} = F_p(\rho)Y$ where $F_p(\rho)$ is given by

$$F_p(\rho) = \begin{pmatrix} \frac{\partial^2 p}{\partial x \partial \xi}(\rho) & \frac{\partial^2 p}{\partial \xi \partial \xi}(\rho) \\ -\frac{\partial^2 p}{\partial x \partial x}(\rho) & -\frac{\partial^2 p}{\partial \xi \partial x}(\rho) \end{pmatrix}$$

and called the Hamilton map (fundamental matrix) of p at ρ .

The special structure of $F_p(\rho)$ results from the fact that $p(x, \xi_0, \xi') = 0$ has only real roots ξ_0 for any (x, ξ') .

Lemma 2.1 ([9, Lemma 9.2, 9.4]). *All eigenvalues of the Hamilton map $F_p(\rho)$ are on the imaginary axis, possibly one exception of a pair of non-zero real eigenvalues.*

We assume that the doubly characteristic set $\Sigma = \{(x, \xi) \mid \partial_\xi^\alpha \partial_x^\beta p(x, \xi) = 0, \forall |\alpha + \beta| < 2\}$ verifies the following conditions:

$$(2.1) \quad \begin{cases} \Sigma \text{ is a } \gamma^{(s)} \text{ manifold,} \\ p \text{ vanishes on } \Sigma \text{ of order exactly 2,} \\ \text{rank}(d\xi \wedge dx) = \text{const. on } \Sigma. \end{cases}$$

Note that $p_\rho(X)$ is *always a strictly hyperbolic polynomial* on $\mathbb{R}^{2(n+1)}/T_\rho\Sigma$ as far as p vanishes on Σ of order exactly 2. We also assume that the codimension Σ is 3 and no transition of spectral type of F_p occur on Σ , that is we assume

$$(2.2) \quad \text{either } \text{Ker } F_p^2 \cap \text{Im } F_p^2 = \{0\} \text{ or } \text{Ker } F_p^2 \cap \text{Im } F_p^2 \neq \{0\}$$

throughout Σ . The following table sums up a general picture of the Gevrey strong hyperbolicity for differential operators with double characteristics ([2, 3, 17, 11]) where $W = \text{Ker } F_p^2 \cap \text{Im } F_p^2$.

Spectrum of F_p	W	Geometry of bicharacteristics near Σ	$G(p)$
Exists non-zero real eigenvalue	$W = \{0\}$	At every point on Σ exactly two bicharacteristics intersect Σ transversally	$G(p) = \infty$
No non-zero real eigenvalue	$W \neq \{0\}$	No bicharacteristic intersects Σ	$G(p) = 4$
		Exists a bicharacteristic tangent to Σ	$G(p) = 3$
	$W = \{0\}$	No bicharacteristic intersects Σ	$G(p) = 2$

This table shows that, assuming (2.1), (2.2) and the codimension Σ is 3, the Gevrey strong hyperbolicity index $G(p)$ takes only the values 2, 3, 4 and ∞ and that these values completely determine the structure of the Hamilton map and the geometry of bicharacteristics near Σ and vice versa.

Lemma 2.2 ([6, Corollary 1.4.7], [12, Lemma 1.1.3]). *Let ρ be a double characteristic. Then the following two conditions are equivalent.*

- (i) $F_p(\rho)$ has non-zero real eigenvalues,
- (ii) $C_\rho \cap T_\rho \Sigma = \{0\}$.

Note that the condition (ii) is well defined for characteristics of *any order* while $F_p(\rho) \equiv 0$ if ρ is a characteristic of order larger than 2.

Remark 2.1. Based on the table, it is quite natural to ask whether the converse of Theorem 1.3 is true. That is if $G(p) = m/(m-2)$ then (1.1) and (1.2) hold?

Remark 2.2. Consider the case $C_\rho \subset T_\rho \Sigma$ that would be considered as a opposite case to $C_\rho \cap T_\rho \Sigma = \{0\}$. Here we note

Lemma 2.3 ([18, Lemma 2.11]). *We have $C_\rho \subset T_\rho \Sigma$ if and only if $T_\rho \Sigma$ is involutive, that is $(T_\rho \Sigma)^\sigma \subset T_\rho \Sigma$ where $(T_\rho \Sigma)^\sigma = \{X \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid (d\xi \wedge dx)(X, Y) = 0, \forall Y \in T_\rho \Sigma\}$.*

It is also natural to ask whether $G(p) = m/(m-1)$ if $C_\rho \subset T_\rho \Sigma$, $\rho \in \Sigma$. When Σ is involutive one can choose homogeneous symplectic coordinates x, ξ in a conic neighborhood of $\rho \in \Sigma$ such that Σ is defined by ([8, Theorem 21.2.4], for example)

$$\xi_0 = \xi_1 = \dots = \xi_k = 0.$$

Thus by conjugation of a Fourier integral operator $p(x, \xi)$ can be written

$$p(x, \xi) = \xi_0^m + \sum_{\alpha_0 \leq m-2, |\alpha|=m} a_\alpha(x, \xi) \tilde{\xi}^\alpha$$

where $\tilde{\xi} = (\xi_0, \xi_1, \dots, \xi_k)$. Thus $\partial_\xi^\alpha \partial_x^\beta p(0, \bar{\xi}) = 0$ for $|\alpha| < m$ and any β . If the resulting $p(x, D)$ is a *differential operator* so that $a_\alpha(x, \xi) = a_\alpha(x)$ then from [10, Theorem 1] we conclude that if the Cauchy problem for $p + P_{m-1} + \dots$ is $\gamma^{(\kappa)}$ well-posed then $\partial_\xi^\alpha \partial_x^\beta P_{m-1}(0, \bar{\xi}) = 0$ for any $|\alpha| \leq m - \kappa/(\kappa - 1)$ and any β . This proves $G(p) \leq m/(m-1)$ and hence $G(p) = m/(m-1)$.

Example 2.1. When $m \geq 3$ the geometry of p with the limit point ρ becomes to be complicated comparing with the case $m = 2$, even (1.1) and (1.2) are satisfied. We give an example. Let us consider

$$p(x, \xi) = \xi_0^3 - 3a\{(x_0^2 + x_1^2)\xi_n^2 + \xi_1^2\}\xi_0 - 2bx_0x_1\xi_1\xi_n^2$$

near $\rho = (0, \dots, 0, 1)$ which is obtained from Example 1.2 with

$$q(\zeta_0, \zeta_1, \zeta_2, \zeta_3) = \zeta_0^3 - 3a(\zeta_1^2 + \zeta_2^2 + \zeta_3^2)\zeta_0 - 2b\zeta_1\zeta_2\zeta_3$$

and $b_1 = x_0\xi_n$, $b_2 = x_1\xi_n$, $b_3 = \xi_1$ where $a > 0$, b are real constants. Choosing $b = \delta a^{3/2}$ with $|\delta| < 1$ and repeating similar arguments as in Example 1.3 it is easily seen that $p(x, \xi)$ satisfies (1.1) and (1.2).

Consider the Hamilton equations

$$(2.3) \quad \dot{x}_j = \partial p / \partial \xi_j, \quad \dot{\xi}_j = -\partial p / \partial x_j, \quad j = 0, \dots, n.$$

Since $\dot{\xi}_n = 0$ we take $\xi_n = 1$ and $x_1 = \xi_0 = 0$ in (2.3) so that the resulting equations reduce to:

$$(2.4) \quad \dot{x}_0 = -3a(x_0^2 + \xi_1^2), \quad \dot{\xi}_1 = 2bx_0\xi_1.$$

We fix $-1 < \delta < 0$ and take $a > 0$ so that $2b/(3a) < -1$. Then any integral curve of (2.4) passing a point in the cone $|\xi_1| < |1 + (2b/3a)|^{1/2}|x_0|$, $x_0 < 0$ arrives at the origin inside the cone (see, for example [20]). In particular there are infinitely many bicharacteristics with the limit point ρ .

3 Preliminaries

Choosing a new system of local coordinates leaving $x_0 = \text{const.}$ to be invariant one can assume that

$$p(x, \xi) = \xi_0^m + a_2(x, \xi')\xi_0^{m-2} + \dots + a_m(x, \xi')$$

and hence $\Sigma \subset \{\xi_0 = 0\}$. Thus near ρ we may assume that Σ is defined by $b_0(x, \xi) = \dots = b_k(x, \xi) = 0$ where $b_0 = \xi_0$, $b_j = b_j(x, \xi')$, $1 \leq j \leq k$ and db_j are linearly independent at ρ' where ρ' stands for (\bar{x}, ξ') when $\rho = (\bar{x}, \xi)$. Recall that the localization $p_\rho(x, \xi)$ is a homogeneous hyperbolic polynomial of degree m in (x, ξ) in the direction $(0, \theta) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$.

Lemma 3.1 ([12, Lemma 1.1.3]). *The next two conditions are equivalent.*

- (i) $C_\rho \cap T_\rho \Sigma = \{0\}$,
- (ii) $\Gamma_\rho \cap (T_\rho \Sigma)^\sigma \cap \langle (0, \theta) \rangle^\sigma \neq \emptyset$.

Assume $C_\rho \cap T_\rho \Sigma = \{0\}$ then thanks to Lemma 3.1 there exists $0 \neq X \in \Gamma_\rho \cap (T_\rho \Sigma)^\sigma \cap \langle (0, \theta) \rangle^\sigma$. Since $(T_\rho \Sigma)^\sigma$ is spanned by $H_{b_j}(\rho)$, $j = 0, \dots, k$ one can write

$$(3.1) \quad X = \sum_{j=0}^k \alpha_j H_{b_j}(\rho)$$

where $\alpha_0 = 0$ because $X \in \langle (0, \theta) \rangle^\sigma$. This proves $\partial_{x_0} b_j(\rho') \neq 0$ with some $1 \leq j \leq k$. Indeed if not we would have $X = (x, 0, \xi')$ while denoting $p_\rho(x, \xi) = \prod_{j=1}^m (\xi_0 - \Lambda_j(x, \xi'))$ we see $\Gamma_\rho = \{(x, \xi) \mid \xi_0 > \max_j \Lambda_j(x, \xi')\}$ (for example [7, Lemma 8.7.3]) and we would have $\Lambda_j(x, \xi') < 0$ which contradicts $\sum_{j=1}^m \Lambda_j(x, \xi') = 0$. Renumbering, if necessary, one can assume $\partial_{x_0} b_1(\rho') \neq 0$ so that

$$b_1(x, \xi') = (x_0 - f_1(x', \xi'))e_1(x, \xi'), \quad e_1(x, \xi') \neq 0.$$

Writing $b_j(x, \xi') = b_j(f_1(x', \xi'), x_1, \dots, x_n, \xi') + c_j(x, \xi')b_1(x, \xi')$ we may assume $b_j(x, \xi')$, $2 \leq j \leq k$ are independent of x_0 . Since $p(x, \xi)$ vanishes on Σ of order m one can write with $b = (b_0, b_1, \dots, b_k) = (b_0, b')$

$$(3.2) \quad p(x, \xi) = b_0^m + \sum_{|\alpha|=m, \alpha_0 \leq m-2} \tilde{a}_\alpha(x, \xi') b(x, \xi)^\alpha.$$

Let \hat{b}_j be defined by $b_j(\rho + \mu X) = \mu \hat{b}_j(X) + O(\mu^2)$ and with $\hat{b} = (\xi_0, \hat{b}_1, \dots, \hat{b}_k)$ we have $p_\rho(X) = q(\hat{b}(X))$ where

$$q(\zeta) = \zeta_0^m + \sum_{|\alpha|=m, \alpha_0 \leq m-2} \tilde{a}_\alpha(\rho') \zeta^\alpha, \quad \zeta = (\zeta_0, \zeta_1, \dots, \zeta_k) = (\zeta_0, \zeta')$$

is a strictly hyperbolic polynomial in the direction $(1, 0, \dots, 0) \in \mathbb{R}^{k+1}$ by (1.1). Denote $\tilde{q}(\zeta; x, \xi') = q(\zeta) + \sum a_\alpha(x, \xi') \zeta^\alpha$ with $a_\alpha(x, \xi') = \tilde{a}_\alpha(x, \xi') - \tilde{a}(\rho')$ and hence we have $p(x, \xi) = \tilde{q}(b(x, \xi); x, \xi')$.

Lemma 3.2. *There are m real valued functions $\lambda_1(x, \xi') \leq \lambda_2(x, \xi') \leq \dots \leq \lambda_m(x, \xi')$ defined in a conic neighborhood of ρ' such that*

$$p(x, \xi) = \prod_{j=1}^m (\xi_0 - \lambda_j(x, \xi')), \quad |\lambda_j(x, \xi')| \leq C|b'(x, \xi')|,$$

$$|\lambda_i(x, \xi') - \lambda_j(x, \xi')| \geq c|b'(x, \xi')|, \quad (i \neq j)$$

with some $c > 0$, $C > 0$.

Proof: The first assertion is clear because $p(x, \xi)$ is a hyperbolic polynomial in the direction ξ_0 . Note that $\tilde{q}(\zeta; \rho') = 0$ has m real distinct roots for $\zeta' \neq 0$ then by Rouché's theorem $\tilde{q}(\zeta_0, \zeta'; x, \xi') = 0$ has m real distinct roots $\zeta_0 = \lambda_j(\zeta'; x, \xi')$ if $|\xi' - \rho'|$ is sufficiently small which are of homogeneous of degree 1 in ζ' and 0 in ξ' . It is easy to check that $|\lambda_j(\zeta'; x, \xi')| \leq C|\zeta'|$ and $|\lambda_i(\zeta'; x, \xi') - \lambda_j(\zeta'; x, \xi')| \geq c|\zeta'|$ ($i \neq j$) with some $c > 0, C > 0$. Since $\{|\zeta'| = 1\}$ is compact we end the proof. \square

4 Basic weights (energy estimates)

We first introduce symbol classes of pseudodifferential operators which will be used in this paper. Denote $\langle \xi \rangle_\gamma^2 = \gamma^2 + |\xi|^2$ where $\gamma \geq 1$ is a positive parameter.

Definition 4.1. Let $W = W(x, \xi; \gamma) > 0$ be a positive function. We define $S_{(s)}(W, g_0)$ to be the set of all $a(x, \xi; \gamma) \in C^\infty(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$ such that

$$(4.1) \quad |\partial_x^\beta \partial_\xi^\alpha a(x, \xi; \gamma)| \leq C A^{|\alpha+\beta|} |\alpha + \beta|!^s W \langle \xi \rangle_\gamma^{-|\alpha|}$$

and $S_{(1)}(W, \bar{g})$ to be the set of all $a(x, \xi; \gamma)$ such that we have

$$(4.2) \quad |\partial_x^\beta \partial_\xi^\alpha a| \leq C A^{|\alpha+\beta|} W (|\alpha + \beta| + |\alpha + \beta|^s \langle \xi \rangle_\gamma^{-\delta/2})^{|\alpha+\beta|} \langle \xi \rangle_\gamma^{-\rho|\alpha|+\delta|\beta|}$$

for any $\alpha, \beta \in \mathbb{N}^{n+1}$ with positive constants $C, A > 0$ independent of $\gamma \geq 1$. If $a(x, \xi; \gamma)$ satisfies (4.1) (resp. (4.2)) in a conic open set $U \subset \mathbb{R}^{n+1} \times (\mathbb{R}^{n+1} \setminus \{0\})$ we say $a(x, \xi; \gamma) \in S_{(s)}(W, g_0)$ (resp. $S_{(1)}(W, \bar{g})$) in U . We often write $a(x, \xi)$ for $a(x, \xi; \gamma)$ dropping γ .

Note that g_0 and \bar{g} is the metric defining the symbol class $S_{1,0}$ and $S_{\rho,\delta}$ respectively. It is clear that one may replace $(|\alpha + \beta| + |\alpha + \beta|^s \langle \xi \rangle_\gamma^{-\delta/2})^{|\alpha+\beta|}$ by $|\alpha + \beta|!(1 + |\alpha + \beta|^{s-1} \langle \xi \rangle_\gamma^{-\delta/2})^{|\alpha+\beta|}$ in (4.2), still defining the same symbol class.

Since $p(x, \xi)$ is a polynomial in ξ of degree m it is clear that $p(x, \xi) \in S_{(s)}(\langle \xi \rangle_\gamma^m, g_0)$. Since $b_j(x, \xi')$ are defined only in a conic neighborhood of $\rho' = (\bar{x}, \bar{\xi}')$ we extend such symbols to $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$. Let $\chi(t) \in \gamma^{(s)}(\mathbb{R})$ be 1 for $|t| < c/2$ and 0 for $|t| > c$ with small $0 < c < 1/2$ and set

$$\begin{cases} y(x) = \chi(|x - \bar{x}|)(x - \bar{x}) + \bar{x}, \\ \eta'(\xi) = \chi(|\xi' \langle \xi \rangle_\gamma^{-1} - \bar{\xi}'|)(\xi' - \langle \xi \rangle_\gamma \bar{\xi}') + \langle \xi \rangle_\gamma \bar{\xi}'. \end{cases}$$

Then it is easy to see $\eta', b_j(y, \eta') \in S_{(s)}(\langle \xi \rangle_\gamma, g_0)$ and $\langle \xi \rangle_\gamma / C \leq |\eta'| \leq C \langle \xi \rangle_\gamma$ with some $C > 0$. In what follows we denote $b_j(y, \eta')$ by $b_j(x, \xi)$.

We now define $w(x, \xi)$, $\omega(x, \xi)$ by

$$\begin{cases} w(x, \xi) = (\sum_{j=1}^k b_j(x, \xi)^2 \langle \xi \rangle_\gamma^{-2} + \langle \xi \rangle_\gamma^{-2\delta})^{1/2}, \\ \omega = (\phi^2 + \langle \xi \rangle_\gamma^{-2\delta})^{1/2}, \quad \phi = \sum_{j=1}^k \alpha_j b_j(x, \xi) \langle \xi \rangle_\gamma^{-1}. \end{cases}$$

Here we recall (3.1), that is

$$(4.3) \quad H_\phi(\rho) \in \Gamma_\rho.$$

In what follows we assume that $0 < \delta < \rho < 1$ verifies $\rho + \delta = 1$ and $1 < s$ satisfies $0 < s - 1 < (1 - \rho)/2\rho$.

Lemma 4.1. *There exist $C, A > 0$ such that*

$$|\partial_x^\beta \partial_\xi^\alpha w| \leq CA^{|\alpha+\beta|} (|\alpha + \beta| + |\alpha + \beta|^s \langle \xi \rangle_\gamma^{-\delta/2})^{|\alpha+\beta|} w \langle \xi \rangle_\gamma^{-\rho|\alpha|+\delta|\beta|}$$

that is $w \in S_{(1)}(w, \bar{g})$. We also have $\omega^{\pm 1} \in S_{(1)}(\omega^{\pm 1}, \bar{g})$.

We first remark an easy lemma.

Lemma 4.2. *Let $M > 0$ be such that $2(1 + 4\sum_{j=0}^\infty (j+1)^{-2})M \leq 1/2$ and $\Gamma_1(k) = Mk!/k^3$, $k \in \mathbb{N}$ where $\Gamma(0) = M$. Then we have*

$$\sum_{\alpha' + \alpha'' = \alpha} \binom{\alpha}{\alpha'} \Gamma_1(|\alpha'|) \Gamma_1(|\alpha''|) \leq \Gamma_1(|\alpha|)/2.$$

Proof of Lemma 4.1: It suffices to prove the assertion for ϵw with small $\epsilon > 0$ so that one can assume $|w| \leq 1$. Thus with $w^2 = F$ there is $A_1 > 0$ such that

$$|\partial_x^\alpha \partial_\xi^\beta F| \leq A_1^{|\alpha+\beta|} \Gamma_1(|\alpha + \beta|) |\alpha + \beta|^{(s-1)|\alpha+\beta|} \langle \xi \rangle_\gamma^{-|\beta|}$$

holds for any α, β . Noting $|\partial_x^\alpha \partial_\xi^\beta w| \leq C_{\alpha\beta} w^{1-|\alpha+\beta|} \langle \xi \rangle_\gamma^{-|\beta|}$ for any α, β we choose $A \geq 2A_1$ so that $C_{\alpha\beta} \leq A^{|\alpha+\beta|} \Gamma_1(|\alpha + \beta|)$ for $|\alpha + \beta| \leq 4$ then we have

$$(4.4) \quad \begin{aligned} |\partial_x^\alpha \partial_\xi^\beta w| &\leq A^{|\alpha+\beta|} \Gamma_1(|\alpha + \beta|) w \langle \xi \rangle_\gamma^{-\rho|\beta|+\delta|\alpha|} \\ &\times (w^{-1} \langle \xi \rangle_\gamma^{-\delta} + |\alpha + \beta|^{s-1} \langle \xi \rangle_\gamma^{-\delta/2})^{|\alpha+\beta|}. \end{aligned}$$

Suppose that (4.4) holds for $|\alpha + \beta| \leq k$, $4 \leq k$ and let $|\alpha + \beta| = k + 1 \geq 4$. Noting

$$2w \partial_x^\alpha \partial_\xi^\beta w = - \sum_{1 \leq |\alpha' + \beta'| \leq k} \binom{\alpha}{\alpha'} \binom{\beta}{\beta'} \partial_x^{\alpha'} \partial_\xi^{\beta'} w \partial_x^{\alpha-\alpha'} \partial_\xi^{\beta-\beta'} w + \partial_x^\alpha \partial_\xi^\beta F$$

and $w^{-1} \geq 1$, applying Lemma 4.2 we see that $w|\partial_x^\alpha \partial_\xi^\beta w|$ is bounded by

$$\begin{aligned} & \frac{1}{2} A^{|\alpha+\beta|} \Gamma_1(|\alpha+\beta|) w^2 \langle \xi \rangle_\gamma^{-\rho|\beta|+\delta|\alpha|} (w^{-1} \langle \xi \rangle_\gamma^{-\delta} + |\alpha+\beta|^{s-1} \langle \xi \rangle_\gamma^{-\delta/2})^{|\alpha+\beta|} \\ & + A_1^{|\alpha+\beta|} \Gamma_1(|\alpha+\beta|) w^2 |\alpha+\beta|^{(s-1)|\alpha+\beta|} (w^{-2} \langle \xi \rangle_\gamma^{-\delta|\alpha+\beta|}) \langle \xi \rangle_\gamma^{-\rho|\beta|+\delta|\alpha|}. \end{aligned}$$

Since we have $w^{-2} \langle \xi \rangle_\gamma^{-\delta|\alpha+\beta|} \leq \langle \xi \rangle_\gamma^{-\delta(|\alpha+\beta|-2)} \leq \langle \xi \rangle_\gamma^{-\delta|\alpha+\beta|/2}$ if $|\alpha+\beta| \geq 4$ then taking $A^{|\alpha+\beta|}/2 + A_1^{|\alpha+\beta|} \leq A^{|\alpha+\beta|}$ into account we conclude that (4.4) holds for $|\alpha+\beta| = k+1$. Therefore noting $w^{-1} \langle \xi \rangle_\gamma^{-\delta} \leq 1$ we get

$$|\partial_x^\alpha \partial_\xi^\beta w| \leq A^{|\alpha+\beta|} \Gamma_1(|\alpha+\beta|) w \langle \xi \rangle_\gamma^{-\rho|\beta|+\delta|\alpha|} (1 + |\alpha+\beta|^{s-1} \langle \xi \rangle_\gamma^{-\delta/2})^{|\alpha+\beta|}.$$

The assertion for ω is proved similarly. As for ω^{-1} , using

$$\begin{aligned} \omega |\partial_x^\alpha \partial_\xi^\beta \omega^{-1}| & \leq \sum \binom{\alpha}{\alpha'} \binom{\beta}{\beta'} A^{|\alpha+\beta|} \Gamma_1(|\alpha'+\beta'|) \Gamma_1(|\alpha+\beta|-|\alpha'+\beta'|) \\ & \quad \times \langle \xi \rangle_\gamma^{-\rho|\beta|+\delta|\alpha|} (w^{-1} \langle \xi \rangle_\gamma^{-\delta} + |\alpha+\beta|^{s-1} \langle \xi \rangle_\gamma^{-\delta/2})^{|\alpha+\beta|} \end{aligned}$$

the proof follows from induction on $|\alpha+\beta|$. \square

We now introduce a basic weight symbol which plays a key role in obtaining energy estimates:

$$\psi = \langle \xi \rangle_\gamma^\kappa \log(\phi + \omega), \quad \kappa = \rho - \delta.$$

Lemma 4.3. *We have $(\phi + \omega)^{\pm 1} \in S_{(1)}((\phi + \omega)^{\pm 1}, \bar{g})$. We have also $\psi \in S_{(1)}(\langle \xi \rangle_\gamma^\kappa \log \langle \xi \rangle_\gamma, \bar{g})$. Moreover $\partial_x^\beta \partial_\xi^\alpha \psi \in S_{(1)}(\omega^{-1} \langle \xi \rangle_\gamma^{\kappa-|\alpha|}, \bar{g})$ for $|\alpha+\beta| = 1$.*

Proof: With $W = \phi + \omega$ we put for $|\alpha+\beta| = 1$

$$(4.5) \quad \partial_x^\beta \partial_\xi^\alpha W = \frac{\partial_x^\beta \partial_\xi^\alpha \phi}{\omega} W + \frac{\partial_x^\beta \partial_\xi^\alpha \langle \xi \rangle_\gamma^{-2\delta}}{2\omega} = \Phi_\beta^\alpha W + \Psi_\beta^\alpha.$$

We examine $\partial_x^\beta \partial_\xi^\alpha \phi \in S_{(1)}(\omega \langle \xi \rangle_\gamma^{-\rho|\alpha|+\delta|\beta|}, \bar{g})$ for $|\alpha+\beta| = 1$. Indeed noting $\omega^{-1} \langle \xi \rangle_\gamma^{-\delta} \leq 1$ we have

$$\begin{aligned} |\partial_x^{\nu+\beta} \partial_\xi^{\mu+\alpha} \phi| & \leq C A^{|\mu+\nu|} \omega |\mu+\nu|^s \langle \xi \rangle_\gamma^{-\delta|\mu+\nu|} \langle \xi \rangle_\gamma^{-\rho|\mu+\alpha|+\delta|\nu+\beta|} \\ & \leq C A^{|\mu+\nu|} \omega \langle \xi \rangle_\gamma^{-\rho|\alpha|+\delta|\beta|} (|\mu+\nu|^s \langle \xi \rangle_\gamma^{-\delta/2})^{|\mu+\nu|} \langle \xi \rangle_\gamma^{-\rho|\mu|+\delta|\nu|}. \end{aligned}$$

Since $\omega^{-1} \in S_{(1)}(\omega^{-1}, \bar{g})$ one can find $A_1 > 0$ such that

$$\begin{aligned} |\partial_x^\nu \partial_\xi^\mu \Phi_\beta^\alpha| & \leq A_1^{|\mu+\nu|+1} \langle \xi \rangle_\gamma^{-\rho|\alpha+\mu|+\delta|\beta+\nu|} |\mu+\nu|! \\ & \quad \times (1 + |\mu+\nu|^{s-1} \langle \xi \rangle_\gamma^{-\delta/2})^{|\mu+\nu|}, \quad \forall \mu, \nu \end{aligned}$$

holds for $|\alpha + \beta| = 1$. Since $\langle \xi \rangle_\gamma^{-2\delta} \leq W$ similar arguments prove

$$(4.6) \quad \begin{aligned} |\partial_x^\nu \partial_\xi^\mu \Psi_\beta^\alpha| &\leq A_1^{|\mu+\nu|+1} W \langle \xi \rangle_\gamma^{-\rho|\alpha+\mu|+\delta|\beta+\nu|} |\mu + \nu|! \\ &\quad \times (1 + |\mu + \nu|^{s-1} \langle \xi \rangle_\gamma^{-\delta/2})^{\mu+\nu}, \quad \forall \mu, \nu. \end{aligned}$$

Now suppose

$$(4.7) \quad \begin{aligned} |\partial_x^\beta \partial_\xi^\alpha W| &\leq C A_2^{|\alpha+\beta|} W \langle \xi \rangle_\gamma^{-\rho|\alpha|+\delta|\beta|} |\alpha + \beta|! \\ &\quad \times (1 + |\alpha + \beta|^{s-1} \langle \xi \rangle_\gamma^{-\delta/2})^{|\alpha+\beta|} \end{aligned}$$

holds for $|\alpha + \beta| \leq \ell$ and letting $|\alpha + \beta + e_1 + e_2| = \ell + 1$ we see

$$\begin{aligned} |\partial_x^{\beta+e_2} \partial_\xi^{\alpha+e_1} W| &\leq C \sum \binom{\alpha}{\alpha'} \binom{\beta}{\beta'} A_1^{|\alpha-\alpha'+\beta-\beta'|+1} A_2^{|\alpha'+\beta'|} \\ &\quad \times |\alpha - \alpha' + \beta - \beta'|! |\alpha' + \beta'|! W \langle \xi \rangle_\gamma^{-\rho|\alpha+e_1|+\delta|\beta+e_2|} \\ &\quad \times (1 + |\alpha + \beta|^{s-1} \langle \xi \rangle_\gamma^{-\delta/2})^{|\alpha+\beta|} \\ &\quad + A_1^{|\alpha+\beta|+1} |\alpha + \beta|! W (1 + |\alpha + \beta|^{s-1} \langle \xi \rangle_\gamma^{-\delta/2})^{|\alpha+\beta|} \langle \xi \rangle_\gamma^{-\rho|\alpha+e_1|+\delta|\beta+e_2|} \\ &\leq (C A_2^{|\alpha+\beta|+1} A_1 (A_2 - A_1)^{-1} + A_1^{|\alpha+\beta|+1}) \\ &\quad \times |\alpha + \beta|! W (1 + |\alpha + \beta|^{s-1} \langle \xi \rangle_\gamma^{-\delta/2})^{|\alpha+\beta|} \langle \xi \rangle_\gamma^{-\rho|\alpha+e_1|+\delta|\beta+e_2|}. \end{aligned}$$

Thus it suffices to choose A_2 so that $A_1(A_2 - A_1)^{-1} + C^{-1}(A_1 A_2^{-1}) \leq 1$ to conclude $\phi + \omega \in S_{(1)}(\phi + \omega, \bar{g})$. As for $(\phi + \omega)^{-1}$ it suffices to repeat the proof of Lemma 4.1.

We turn to the next assertion. From $\langle \xi \rangle_\gamma^{-2\delta}/C \leq \phi + \omega \leq C$ it is clear $|\psi| \leq \langle \xi \rangle_\gamma^\kappa \log \langle \xi \rangle_\gamma$. Since $\partial_x^\beta \partial_\xi^\alpha \log(\phi + \omega) = \partial_x^\beta \partial_\xi^\alpha (\phi + \omega)/(\phi + \omega)$ for $|\alpha + \beta| = 1$ and $(\phi + \omega)^{\pm 1} \in S_{(1)}((\phi + \omega)^{\pm 1}, \bar{g})$ we see $\psi \in S_{(1)}(\langle \xi \rangle_\gamma^\kappa \log \langle \xi \rangle_\gamma, \bar{g})$. Since $\partial_x^\beta \partial_\xi^\alpha \phi \in S_{(1)}(\langle \xi \rangle_\gamma^{-|\alpha|}, \bar{g})$ for $|\alpha + \beta| = 1$ and $\omega^{-1} \in S_{(1)}(\omega^{-1}, \bar{g})$ it follows from (4.5) and (4.7) that

$$\begin{aligned} |\partial_x^\nu \partial_\xi^\mu (\partial_x^\beta \partial_\xi^\alpha W)| &\leq C A^{|\mu+\nu|} \omega^{-1} W \langle \xi \rangle_\gamma^{-|\alpha|} |\mu + \nu|! \\ &\quad \times (1 + |\mu + \nu|^{s-1} \langle \xi \rangle_\gamma^{-\delta/2})^{\mu+\nu} \langle \xi \rangle_\gamma^{-\rho|\mu|+\delta|\nu|} \end{aligned}$$

which proves the second assertion. \square

5 Composition formula (energy estimates)

In studying $\text{Op}(e^\psi) P \text{Op}(e^{-\psi}) = \text{Op}(e^\psi \# P \# e^{-\psi})$, if $\psi \in S_{\rho, \delta}^{\kappa'}$ with $\kappa' < \rho - \delta$ then one can apply the calculus obtained in [19] to get an asymptotic formula

of $e^\psi \# P \# e^{-\psi}$, where the proof is based on the almost analytic extension of symbols and the Stokes' formula using a space $\rho - \delta - \kappa' > 0$. In the present case $\psi \in S_{\rho, \delta}^\kappa$ there is no space between κ and $\rho - \delta$ and then, introducing a small parameter $\epsilon > 0$, we carefully estimate $e^{\epsilon\psi} \# p \# e^{-\epsilon\psi}$ directly to obtain the composition formula in Theorem 5.1 below.

We denote $a(x, \xi; \gamma, \epsilon) \in \epsilon^{\kappa_1} S_{(1)}(W, \bar{g})$ if $\epsilon^{-\kappa_1} a \in S_{(1)}(W, \bar{g})$ uniformly in $0 < \epsilon \ll 1$. Our aim in this section is to give a sketch of the proof of

Theorem 5.1. *Let $p(x, \xi) \in S_{(s)}(\langle \xi \rangle_\gamma^m, g_0)$. Then there exists $\epsilon_0 > 0$ such that one can find $K = 1 + r$, $r \in \sqrt{\epsilon} S(1, \bar{g})$ and $\gamma_0(\epsilon) > 0$ for $0 < \epsilon \leq \epsilon_0$ so that we have*

$$\begin{aligned} e^{\epsilon\psi} \# p \# e^{-\epsilon\psi} \# K &= \sum_{|\alpha+\beta| \leq m} \frac{\epsilon^{|\alpha+\beta|}}{\alpha! \beta!} p_{(\alpha)}^{(\beta)} (-i \nabla_\xi \psi)^\alpha (i \nabla_x \psi)^\beta \\ &\quad + \sum_{1 \leq |\alpha+\beta| \leq m} \epsilon^{|\alpha+\beta|+1/2} c_\beta^\alpha p_{(\alpha)}^{(\beta)} + R \end{aligned}$$

where $p_{(\alpha)}^{(\beta)} = \partial_\xi^\beta \partial_x^\alpha p$ and $c_\beta^\alpha \in S(\langle \xi \rangle_\gamma^{\rho|\beta|-\delta|\alpha|}, \bar{g})$, $R \in S(\langle \xi \rangle_\gamma^{m-\delta(m+1)}, \bar{g})$. Moreover we have $c_\beta^\alpha \in S(\omega^{-1} \langle \xi \rangle_\gamma^{\kappa-|\alpha|}, \bar{g})$ for $|\alpha + \beta| = 1$. In particular $e^{\epsilon\psi} \# p \# e^{-\epsilon\psi} \in S(\langle \xi \rangle_\gamma^m, \bar{g})$.

5.1 Estimates of symbol $e^{\epsilon\psi}$

Let $H = H(x, \xi; \gamma) > 0$ be a positive function. Assume that f satisfies

$$\begin{aligned} |\partial_x^\nu \partial_\xi^\mu f| &\leq C_0 A_0^{|\mu+\nu|} (|\mu + \nu| - 1)! \\ &\quad \times (1 + (|\mu + \nu| - 1)^{s-1} \langle \xi \rangle_\gamma^{-\delta/2})^{|\mu+\nu|-1} H \langle \xi \rangle_\gamma^{\delta|\nu|-\rho|\mu|} \end{aligned}$$

for $|\mu + \nu| \geq 1$. Set $\Omega_\beta^\alpha = e^{-f} \partial_x^\beta \partial_\xi^\alpha e^f$ then we have

Lemma 5.1. *Notations being as above. There exist $A_i, C > 0$ such that the following estimate holds for $|\alpha + \beta| \geq 1$;*

$$\begin{aligned} |\partial_x^\nu \partial_\xi^\mu \Omega_\beta^\alpha| &\leq C A_1^{|\nu+\mu|} A_2^{|\alpha+\beta|} \langle \xi \rangle_\gamma^{\delta|\beta+\nu|-\rho|\alpha+\mu|} \\ &\quad \times \sum_{j=1}^{|\alpha+\beta|} H^{|\alpha+\beta|-j+1} (|\mu + \nu| + j)! (1 + (|\mu + \nu| + j)^{s-1} \langle \xi \rangle_\gamma^{-\delta/2})^{|\mu+\nu|+j}. \end{aligned}$$

Corollary 5.1. *We have with some $A, C > 0$*

$$|\partial_x^\beta \partial_\xi^\alpha e^f| \leq C A^{|\alpha+\beta|} \langle \xi \rangle_\gamma^{\delta|\beta|-\rho|\alpha|} (H + |\alpha + \beta| + |\alpha + \beta|^s \langle \xi \rangle_\gamma^{-\delta/2})^{|\alpha+\beta|} e^f.$$

Moreover for $|\alpha + \beta| \geq 1$, $|\partial_x^\beta \partial_\xi^\alpha e^f|$ is bounded by

$$CHA^{|\alpha+\beta|} \langle \xi \rangle_\gamma^{\delta|\beta|-\rho|\alpha|} (H + |\alpha + \beta| + |\alpha + \beta|^s \langle \xi \rangle_\gamma^{-\delta/2})^{|\alpha+\beta|-1} e^f.$$

Corollary 5.2. *Notations being as above. We have for $|\alpha + \beta| \geq 1$*

$$\Omega_\beta^\alpha \in S_{(1)}(H(H + |\alpha + \beta| + |\alpha + \beta|^s \langle \xi \rangle_\gamma^{-\delta/2})^{|\alpha+\beta|-1} \langle \xi \rangle_\gamma^{-\rho|\alpha|+\delta|\beta|}, \bar{g}).$$

Corollary 5.3. *Let $\omega_\beta^\alpha = e^{-\epsilon\psi} \partial_x^\beta \partial_\xi^\alpha e^{\epsilon\psi}$. Then there exists $\gamma_0(\epsilon) > 0$ such that $\omega_\beta^\alpha \in \epsilon^{|\alpha+\beta|} S_{(1)}(\langle \xi \rangle_\gamma^{\rho|\beta|-\delta|\alpha|}, \bar{g})$ for $\gamma \geq \gamma_0(\epsilon)$.*

5.2 Estimates of $(be^{\epsilon\psi})\#e^{-\epsilon\psi}$

Let $\chi(r) \in \gamma^{(s)}(\mathbb{R})$ be 1 in $|r| \leq 1/4$ and 0 outside $|r| \leq 1/2$. Let $b \in S_{(1)}(\omega^t \langle \xi \rangle_\gamma^m, \bar{g})$ and consider

$$\begin{aligned} (be^{\epsilon\psi})\#e^{-\epsilon\psi} &= \int e^{-2i(\eta z - y\zeta)} b(X+Y) e^{\epsilon\psi(X+Y) - \epsilon\psi(X+Z)} dY dZ \\ &= b(X) + \int e^{-2i(\eta z - y\zeta)} b(X+Y) (e^{\epsilon\psi(X+Y) - \epsilon\psi(X+Z)} - 1) dY dZ \end{aligned}$$

where $Y = (y, \eta)$, $Z = (z, \zeta)$. Denoting $\hat{\chi} = \chi(\langle \eta \rangle \langle \xi \rangle_\gamma^{-1}) \chi(\langle \zeta \rangle \langle \xi \rangle_\gamma^{-1})$, $\tilde{\chi} = \chi(|y|/4) \chi(|z|/4)$ we write

$$\begin{aligned} &\int e^{-2i(\eta z - y\zeta)} b(X+Y) (e^{\epsilon\psi(X+Y) - \epsilon\psi(X+Z)} - 1) \{\tilde{\chi}\hat{\chi} + (1 - \tilde{\chi})\hat{\chi}\} dY dZ \\ &+ \int e^{-2i(\eta z - y\zeta)} b(X+Y) (e^{\epsilon\psi(X+Y) - \epsilon\psi(X+Z)} - 1) (1 - \hat{\chi}) dY dZ. \end{aligned}$$

After the change of variables $Z \rightarrow Z + Y$ the first integral turns to

$$(5.1) \quad \int e^{-2i(\eta z - y\zeta)} b(X+Y) (e^{\epsilon\psi(X+Y) - \epsilon\psi(X+Y+Z)} - 1) \hat{\chi}_0 dY dZ$$

where we have set $\hat{\chi}_0 = \tilde{\chi}(y, z) \chi(\langle \eta \rangle \langle \xi \rangle_\gamma^{-1}) \chi(\langle \eta + \zeta \rangle \langle \xi \rangle_\gamma^{-1})$.

Lemma 5.2. *Let $\Psi(X, Y, Z) = \psi(X+Y) - \psi(X+Y+Z)$ then on the support of $\hat{\chi}_0$ one has*

$$|\Psi(X, Y, Z)| \leq C \langle \xi \rangle_\gamma^\kappa \bar{g}_X^{1/2}(Z).$$

On the support of $\hat{\chi}_0$ we have for $|\alpha + \beta| \geq 1$

$$\begin{aligned} |\partial_{(x,y)}^\beta \partial_{(\xi,\eta)}^\alpha e^{\epsilon\Psi}| &\leq \epsilon C A^{|\alpha+\beta|} \langle \xi \rangle_\gamma^{-\rho|\alpha|+\delta|\beta|} \langle \xi \rangle_\gamma^\kappa \bar{g}_X^{1/2}(Z) \\ &\times (\epsilon \langle \xi \rangle_\gamma^\kappa \bar{g}_X^{1/2}(Z) + |\alpha + \beta| + |\alpha + \beta|^s \langle \xi \rangle_\gamma^{-\delta/2})^{|\alpha+\beta|-1} e^{\epsilon\Psi}. \end{aligned}$$

Proof: The assertions follow from Lemma 1.1 and Corollary 5.2. \square

Introducing the following differential operators and symbols

$$\begin{cases} L = 1 + 4^{-1} \langle \xi \rangle_\gamma^{2\rho} |D_\eta|^2 + 4^{-1} \langle \xi \rangle_\gamma^{-2\delta} |D_y|^2, \\ M = 1 + 4^{-1} \langle \xi \rangle_\gamma^{2\delta} |D_\zeta|^2 + 4^{-1} \langle \xi \rangle_\gamma^{-2\rho} |D_z|^2, \\ \Phi = 1 + \langle \xi \rangle_\gamma^{2\rho} |z|^2 + \langle \xi \rangle_\gamma^{-2\delta} |\zeta|^2 = 1 + \langle \xi \rangle_\gamma^{2\kappa} \bar{g}_X(Z), \\ \Theta = 1 + \langle \xi \rangle_\gamma^{2\delta} |y|^2 + \langle \xi \rangle_\gamma^{-2\rho} |\eta|^2 = 1 + \bar{g}_X(Y) \end{cases}$$

so that $\Phi^{-N} L^N e^{-2i(\eta z - y \zeta)} = e^{-2i(\eta z - y \zeta)}$, $\Theta^{-\ell} M^\ell e^{-2i(\eta z - y \zeta)} = e^{-2i(\eta z - y \zeta)}$ we make integration by parts in (5.1). Let $F = b(X + Y)(e^{\epsilon\Psi} - 1)$, $\chi^* = \chi(\epsilon\Phi)$, $\chi_* = 1 - \chi^*$ and note $|(\langle \xi \rangle_\gamma^{-\rho} \partial_z)^\alpha (\langle \xi \rangle_\gamma^\delta \partial_\zeta)^\beta \chi^*| \leq C_{\alpha\beta}$ with $C_{\alpha\beta}$ independent of ϵ . Consider

$$(5.2) \quad \begin{aligned} & \int e^{-2i(\eta z - y \zeta)} \chi^* \partial_x^\beta \partial_\xi^\alpha F \hat{\chi}_0 dY dZ \\ &= \int e^{-2i(\eta z - y \zeta)} \Phi^{-N} L^N \Theta^{-\ell} M^\ell (\chi^* \partial_x^\beta \partial_\xi^\alpha F \hat{\chi}_0) dY dZ. \end{aligned}$$

Applying Corollary 5.1 we can estimate the integrand of the right-hand side of (5.2);

$$\begin{aligned} & |\Phi^{-N} L^N \Theta^{-\ell} M^\ell (\chi^* \partial_x^\beta \partial_\xi^\alpha F \hat{\chi}_0)| \leq C_\ell A^{2N+|\alpha+\beta|+\ell} \Phi^{-N} \Theta^{-\ell} \{ \epsilon \langle \xi \rangle_\gamma^\kappa g_X^{1/2}(Z) \\ & \times (\epsilon \langle \xi \rangle_\gamma^\kappa g_X^{1/2}(Z) + 2N + |\alpha + \beta| + (2N + |\alpha + \beta|)^s \langle \xi \rangle_\gamma^{-\delta/2})^{2N+|\alpha+\beta|-1} e^{\epsilon\Psi} \\ & + |e^{\epsilon\Psi} - 1| (2N + |\alpha + \beta| + (2N + |\alpha + \beta|)^s \langle \xi \rangle_\gamma^{-\delta/2})^{2N+|\alpha+\beta|} \} \\ & \quad \times \omega^t(X + Y) \langle \xi \rangle_\gamma^m. \end{aligned}$$

Here we remark the following easy lemma.

Lemma 5.3. *Let $A \geq 0$, $B \geq 0$. Then there exists $C > 0$ independent of $n, m \in \mathbb{N}$, A, B such that*

$$(A + n + m + (n + m)^s B)^{n+m} \leq C^{n+m} (A + n + n^s B)^n (A + m + m^s B)^m.$$

Since $|e^{\epsilon\Psi} - 1| \leq C|\epsilon\Psi| \leq C\epsilon\Phi^{1/2} \leq C\sqrt{\epsilon}$ on the support of χ^* , the right-hand side can be estimated by

$$\begin{aligned} & C_\ell A_1^{2N+|\alpha+\beta|} \Phi^{-N} \Theta^{-\ell} \{ \epsilon \langle \xi \rangle_\gamma^\kappa \bar{g}_X^{1/2}(Z) (\epsilon \langle \xi \rangle_\gamma^\kappa \bar{g}_X^{1/2}(Z) + 2N - 1 \\ & + (2N - 1)^s \langle \xi \rangle_\gamma^{-\delta/2})^{2N-1} (\epsilon \langle \xi \rangle_\gamma^\kappa \bar{g}_X^{1/2}(Z) + |\alpha + \beta| \\ & + |\alpha + \beta|^s \langle \xi \rangle_\gamma^{-\delta/2})^{|\alpha+\beta|-1} + \sqrt{\epsilon} (\epsilon \langle \xi \rangle_\gamma^\kappa \bar{g}_X^{1/2}(Z) + 2N + (2N)^s \langle \xi \rangle_\gamma^{-\delta/2})^{2N} \\ & \times (\epsilon \langle \xi \rangle_\gamma^\kappa \bar{g}_X^{1/2}(Z) + |\alpha + \beta| + |\alpha + \beta|^s \langle \xi \rangle_\gamma^{-\delta/2})^{|\alpha+\beta|} \} \omega^t(X + Y) \langle \xi \rangle_\gamma^m \end{aligned}$$

where we remark $\omega^{\pm 1}(X + Y) \leq C\omega^{\pm 1}(X)(1 + \bar{g}_X^{1/2}(Y))$ on the support of $\hat{\chi}_0$ and hence we have $\omega^t(X + Y) \leq C\omega^t(X)\Theta^{t'}$ with some t' . Noting

$$\begin{aligned} & A_1^{2N} \Phi^{-N} (\epsilon \langle \xi \rangle_\gamma^\kappa \bar{g}_X^{1/2}(Z) + 2N + (2N)^s \langle \xi \rangle_\gamma^{-\delta/2})^{2N} \\ &= \left(\epsilon \frac{A_1 \langle \xi \rangle_\gamma^\kappa \bar{g}_X^{1/2}(Z)}{\Phi^{1/2}} + \frac{2A_1 N}{\Phi^{1/2}} + \frac{A_1 (2N)^s \langle \xi \rangle_\gamma^{-\delta/2}}{\Phi^{1/2}} \right)^{2N} \end{aligned}$$

and

$$\begin{aligned} & A_1^{2N} \Phi^{-N} \epsilon \langle \xi \rangle_\gamma^\kappa \bar{g}_X^{1/2}(Z) (\epsilon \langle \xi \rangle_\gamma^\kappa \bar{g}_X^{1/2}(Z) + 2N - 1 + (2N - 1)^s \langle \xi \rangle_\gamma^{-\delta/2})^{2N-1} \\ & \leq \epsilon A_1 \left(\epsilon \frac{A_1 \langle \xi \rangle_\gamma^\kappa \bar{g}_X^{1/2}(Z)}{\Phi^{1/2}} + \frac{2A_1 N}{\Phi^{1/2}} + \frac{A_1 (2N)^s \langle \xi \rangle_\gamma^{-\delta/2}}{\Phi^{1/2}} \right)^{2N-1} \end{aligned}$$

we choose $N = N(z, \zeta, \xi)$ so that $2A_1 N = \bar{c} \Phi^{1/2}$ with a small $\bar{c} > 0$. Then noting that $\Phi \leq 1 + 2\langle \xi \rangle_\gamma^{2\rho} |z|^2 + 2\langle \xi \rangle_\gamma^{2\delta} |\zeta|^2 \leq C\langle \xi \rangle_\gamma^{2\rho}$ on the support of $\hat{\chi}_0$ and therefore $\Phi^{(s-1)/2} \langle \xi \rangle_\gamma^{-\delta/2} \leq C\langle \xi \rangle_\gamma^{-\epsilon_1} \leq C\gamma^{-\epsilon_1}$ with some $\epsilon_1 > 0$ we have

$$\begin{aligned} & \left(\epsilon \frac{A_1 \langle \xi \rangle_\gamma^\kappa \bar{g}_X^{1/2}(Z)}{\Phi^{1/2}} + \frac{2A_1 N}{\Phi^{1/2}} + \frac{A_1 (2N)^s \langle \xi \rangle_\gamma^{-\delta/2}}{\Phi^{1/2}} \right)^{2N-1} \\ & \leq (A_1 \epsilon + \bar{c} + \bar{c}^s \Phi^{(s-1)/2} \langle \xi \rangle_\gamma^{-\delta/2})^{2N-1} \leq C e^{-c_1 \Phi^{1/2}} \end{aligned}$$

choosing \bar{c} small and $\gamma \geq \gamma_0(\epsilon)$ large. On the other hand since $\langle \xi \rangle_\gamma^\kappa \bar{g}_X^{1/2}(Z) \leq \Phi^{1/2}$ it is clear

$$\begin{aligned} & (\epsilon \langle \xi \rangle_\gamma^\kappa \bar{g}_X^{1/2}(Z) + |\alpha + \beta| + |\alpha + \beta|^s \langle \xi \rangle_\gamma^{-\delta/2})^{|\alpha + \beta|} e^{-c \Phi^{1/2}} \\ & \leq C A^{|\alpha + \beta|} (|\alpha + \beta| + |\alpha + \beta|^s \langle \xi \rangle_\gamma^{-\delta/2})^{|\alpha + \beta|} e^{-c' \Phi^{1/2}}. \end{aligned}$$

Set $\ell' = \ell - t'$. Then noting $e^{-c' \Phi^{1/2}} \leq C \Phi^{-\ell'}$ we have

$$\begin{aligned} & |\Phi^{-N} L^N \Theta^{-\ell} M^\ell (\chi^* \partial_x^\beta \partial_\xi^\alpha F \hat{\chi}_0)| \\ (5.3) \quad & \leq \sqrt{\epsilon} C A^{|\alpha + \beta|} (|\alpha + \beta| + |\alpha + \beta|^s \langle \xi \rangle_\gamma^{-\delta/2})^{|\alpha + \beta|} \\ & \quad \times \omega^t(X) \langle \xi \rangle_\gamma^{m - \rho|\alpha| + \delta|\beta|} \Theta^{-\ell'} \Phi^{-\ell'}. \end{aligned}$$

Finally choosing $\ell > t' + (n + 1)/2$ and recalling $\int \Theta^{-\ell'} \Phi^{-\ell'} dY dZ = C$ we conclude

$$\begin{aligned} & \left| \int e^{-2i(\eta z - y \zeta)} \chi^* \partial_x^\beta \partial_\xi^\alpha F \hat{\chi}_0 dY dZ \right| \\ & \leq \sqrt{\epsilon} C A^{|\alpha + \beta|} (|\alpha + \beta| + |\alpha + \beta|^s \langle \xi \rangle_\gamma^{-\delta/2})^{|\alpha + \beta|} \omega^t(X) \langle \xi \rangle_\gamma^{m - \rho|\alpha| + \delta|\beta|}. \end{aligned}$$

We next consider

$$\begin{aligned} & \int e^{-2i(\eta z - y\zeta)} \chi_* \partial_x^\beta \partial_\xi^\alpha F \hat{\chi}_0 dY dZ \\ &= \int e^{-2i(\eta z - y\zeta)} L^N \Phi^{-N} M^\ell \Theta^{-\ell} \chi_* \partial_x^\beta \partial_\xi^\alpha F \hat{\chi}_0 dY dZ. \end{aligned}$$

Similar arguments obtaining (5.3) show that

$$\begin{aligned} |\Phi^{-N} L^N \Theta^{-\ell} M^\ell (\chi_* \partial_x^\beta \partial_\xi^\alpha F \hat{\chi}_0)| &\leq C A^{|\alpha+\beta|} (|\alpha+\beta| + |\alpha+\beta|^s \langle \xi \rangle_\gamma^{-\delta/2})^{|\alpha+\beta|} \\ &\quad \times \omega^t(X) \langle \xi \rangle_\gamma^{m-\rho|\alpha|+\delta|\beta|} \Theta^{-\ell'} \Phi^{-\ell'} e^{-c\Phi^{1/2}}. \end{aligned}$$

Since $\Phi^{1/2} \geq \epsilon^{-1/2}$ on the support of χ_* we see $e^{-c\Phi^{1/2}} \leq e^{-c\epsilon^{-1/2}} \leq C\sqrt{\epsilon}$ and this proves

$$\begin{aligned} & \left| \partial_x^\beta \partial_\xi^\alpha \int e^{-2i(\eta z - y\zeta)} F \hat{\chi}_0 dY dZ \right| \\ & \leq \sqrt{\epsilon} C A^{|\alpha+\beta|} (|\alpha+\beta| + |\alpha+\beta|^s \langle \xi \rangle_\gamma^{-\delta/2})^{|\alpha+\beta|} \omega^t(X) \langle \xi \rangle_\gamma^{m-\rho|\alpha|+\delta|\beta|}. \end{aligned}$$

We then consider

$$\int e^{-2i(\eta z - y\zeta)} (|y|^2 + |z|^2)^{-N} (|D_\zeta|^2 + |D_\eta|^2)^N F \hat{\chi}_1 dY dZ$$

where $F = b(X+Y)(e^{\epsilon\psi(X+Y)-\epsilon\psi(X+Z)} - 1)$ and $\hat{\chi}_1 = (1 - \tilde{\chi})\hat{\chi}$. Let $\kappa < \kappa_1 < \rho$ then since $|\psi(X+Y)| + |\psi(X+Z)|$ is bounded by $C\langle \xi \rangle_\gamma^{\kappa_1}$ and $\langle \xi + \eta \rangle_\gamma \approx \langle \xi \rangle_\gamma$, $\langle \xi + \zeta \rangle_\gamma \approx \langle \xi \rangle_\gamma$ on the support of $\hat{\chi}_1$ thanks to Corollary 5.1 it is not difficult to show

$$\begin{aligned} & (|D_\zeta|^2 + |D_\eta|^2)^N \partial_x^\beta \partial_\xi^\alpha F \hat{\chi}_1 \leq C A^{2N+|\alpha+\beta|} \omega^t(X+Y) \langle \xi \rangle_\gamma^{m-\rho|\alpha|+\delta|\beta|} \\ & \quad \times (\langle \xi \rangle_\gamma^{\kappa_1} + |\alpha+\beta| + |\alpha+\beta|^s \langle \xi \rangle_\gamma^{-\delta/2})^{|\alpha+\beta|} \\ & \quad \times \langle \xi \rangle_\gamma^{-2\rho N} (\langle \xi \rangle_\gamma^{\kappa_1} + 2N + (2N)^s \langle \xi \rangle_\gamma^{-\delta/2})^{2N} e^{c\langle \xi \rangle_\gamma^{\kappa_1}}. \end{aligned}$$

Choose $N = c_1 \langle \xi \rangle_\gamma^\rho$ with small $c_1 > 0$ so that

$$A^{2N} \langle \xi \rangle_\gamma^{-2\rho N} (\langle \xi \rangle_\gamma^{\kappa_1} + 2N + (2N)^s \langle \xi \rangle_\gamma^{-\delta/2})^{2N}$$

is bounded by $Ce^{-c\langle \xi \rangle_\gamma^\rho}$ and $\langle \xi \rangle_\gamma^{\kappa_1|\alpha+\beta|} e^{-c\langle \xi \rangle_\gamma^\rho}$ is bounded by $CA^{|\alpha+\beta|} e^{-c'\langle \xi \rangle_\gamma^\rho}$. Then noting $\omega^t(X+Y) \leq C\omega^t(X)\langle \xi \rangle_\gamma^{t'}$ and $e^{-c'\langle \xi \rangle_\gamma^\rho} \leq \sqrt{\epsilon} C \langle \xi \rangle_\gamma^{-2(n+1)-t'}$ for $\gamma \geq \gamma_0(\epsilon)$ and that $\langle \xi \rangle_\gamma^{-2(n+1)} \int (|y|^2 + |z|^2)^{-N} \hat{\chi}_1 dY dZ \leq C$ we conclude

Lemma 5.4. *Let $\hat{\chi} = \chi(\langle \eta \rangle \langle \xi \rangle_\gamma^{-1}) \chi(\langle \zeta \rangle \langle \xi \rangle_\gamma^{-1})$. Then we have for $\gamma \geq \gamma_0(\epsilon)$*

$$\begin{aligned} & \left| \partial_x^\beta \partial_\xi^\alpha \int e^{-2i(z\eta - y\zeta)} b(X+Y) (e^{\epsilon\psi(X+Y) - \epsilon\psi(X+Z)} - 1) \hat{\chi} dY dZ \right| \\ & \leq \sqrt{\epsilon} C A^{|\alpha+\beta|} (|\alpha+\beta| + |\alpha+\beta|^s \langle \xi \rangle_\gamma^{-\delta/2})^{|\alpha+\beta|} \omega^t(X) \langle \xi \rangle_\gamma^{m-\rho|\alpha|+\delta|\beta|}. \end{aligned}$$

Let us write

$$\begin{aligned} 1 - \hat{\chi} &= (1 - \chi(\langle \eta \rangle \langle \xi \rangle_\gamma^{-1})) (1 - \chi(\langle \zeta \rangle \langle \xi \rangle_\gamma^{-1})) + (1 - \chi(\langle \eta \rangle \langle \xi \rangle_\gamma^{-1})) \chi(\langle \zeta \rangle \langle \xi \rangle_\gamma^{-1}) \\ & \quad + (1 - \chi(\langle \zeta \rangle \langle \xi \rangle_\gamma^{-1})) \chi(\langle \eta \rangle \langle \xi \rangle_\gamma^{-1}) = \hat{\chi}_2 + \hat{\chi}_3 + \hat{\chi}_4. \end{aligned}$$

Denoting $F = b(X+Y) (e^{\epsilon\psi(X+Y) - \epsilon\psi(X+Z)} - 1)$ again we consider

$$\begin{aligned} & \int e^{-2i(\eta z - y\zeta)} \langle \eta \rangle^{-2N_2} \langle \zeta \rangle^{-2N_1} \langle D_z \rangle^{2N_2} \langle D_y \rangle^{2N_1} \\ & \times \langle y \rangle^{-2\ell} \langle z \rangle^{-2\ell} \langle D_\zeta \rangle^{2\ell} \langle D_\eta \rangle^{2\ell} \partial_x^\beta \partial_\xi^\alpha F \hat{\chi}_2 \chi^* dY dZ \end{aligned}$$

where χ^* is either $\chi(\langle \zeta \rangle \langle \eta \rangle^{-1}/4)$ or $1 - \chi(\langle \zeta \rangle \langle \eta \rangle^{-1}/4)$. If $\chi^* = \chi(\langle \zeta \rangle \langle \eta \rangle^{-1}/4)$ we choose $N_1 = \ell$, $N_2 = N$ and noting $\omega^t(X+Y) \leq C \langle \eta \rangle^{t'}$ with some $t' \geq 0$ and $|\psi(X+Y)| + |\psi(X+Z)| \leq C \langle \eta \rangle^{\kappa_1}$ with $\kappa < \kappa_1 < \rho$ on the support of $\hat{\chi}_2 \chi^*$ it is not difficult to see that

$$|\langle \eta \rangle^{-2N} \langle \zeta \rangle^{-2\ell} \langle D_z \rangle^{2N} \langle D_y \rangle^{2\ell} \langle y \rangle^{-2\ell} \langle z \rangle^{-2\ell} \langle D_\zeta \rangle^{2\ell} \langle D_\eta \rangle^{2\ell} \partial_x^\beta \partial_\xi^\alpha F \hat{\chi}_2 \chi^*|$$

is bounded by

$$\begin{aligned} & C_\ell A^{2N+|\alpha+\beta|} \langle \eta \rangle^{-2N} \langle \zeta \rangle^{-2\ell} \langle y \rangle^{-2\ell} \langle z \rangle^{-2\ell} \langle \eta \rangle^{m+t'+2\delta\ell} \langle \eta \rangle^{6\ell\rho} \\ (5.4) \quad & \times (\langle \eta \rangle^{\kappa_1} + 2N \langle \eta \rangle^\delta + N^s \langle \eta \rangle^{\delta/2})^{2N} \\ & \times (\langle \eta \rangle^{\kappa_1} + \langle \eta \rangle^\delta |\alpha+\beta| + \langle \eta \rangle^{\delta/2} |\alpha+\beta|^s)^{|\alpha+\beta|} e^{C \langle \eta \rangle^{\kappa_1}}. \end{aligned}$$

Here writing

$$\begin{aligned} & A^{2N} \langle \eta \rangle^{-2N} (\langle \eta \rangle^{\kappa_1} + 2N \langle \eta \rangle^\delta + N^s \langle \eta \rangle^{\delta/2})^{2N} \\ & = \left(\frac{A \langle \eta \rangle^{\kappa_1}}{\langle \eta \rangle} + \frac{2AN}{\langle \eta \rangle^\rho} + \frac{AN^s \langle \eta \rangle^{\delta/2}}{\langle \eta \rangle} \right)^{2N} \end{aligned}$$

we take $2N = c_1 \langle \eta \rangle^\rho$ with small $c_1 > 0$ so that the right-hand side is bounded by $C e^{-c \langle \eta \rangle^\rho}$. Noting $\langle \eta \rangle^{\delta|\alpha+\beta|} e^{-c \langle \eta \rangle^\rho} \leq C A_1^{|\alpha+\beta|} |\alpha+\beta|^{\delta|\alpha+\beta|/\rho} e^{-c_1 \langle \eta \rangle^\rho}$ and $\langle \eta \rangle^{\kappa_1|\alpha+\beta|} e^{-c \langle \eta \rangle^\rho} \leq C A_1^{|\alpha+\beta|} |\alpha+\beta|^{|\alpha+\beta|} e^{-c_1 \langle \eta \rangle^\rho}$ one sees that (5.4) is bounded by

$$C_\ell A_1^{|\alpha+\beta|} \langle \zeta \rangle^{-2\ell} \langle y \rangle^{-2\ell} \langle z \rangle^{-2\ell} (|\alpha+\beta|^{1+\delta/\rho} + |\alpha+\beta|^{s+\delta/2\rho})^{|\alpha+\beta|} e^{-c_1 \langle \eta \rangle^\rho}.$$

Similarly if $\chi^* = 1 - \chi(\langle \zeta \rangle \langle \eta \rangle^{-1}/4)$ choosing $N_1 = N$, $N_2 = \ell$ it is proved that (5.4) is estimated by

$$C_\ell A_1^{|\alpha+\beta|} \langle \eta \rangle^{-2\ell} \langle y \rangle^{-2\ell} \langle z \rangle^{-2\ell} (|\alpha + \beta|^{1+\delta/\rho} + |\alpha + \beta|^{s+\delta/2\rho}) |\alpha + \beta| e^{-c_1 \langle \zeta \rangle^\rho}.$$

Thus taking $1 + \delta/\rho = 1/\rho$ and $s + \delta/2\rho \leq 1/\rho$ into account and recalling that $\langle \xi \rangle_\gamma \leq \langle \eta \rangle$, $\langle \xi \rangle_\gamma \leq \langle \zeta \rangle$ on the support of $\hat{\chi}_2$ we get

Lemma 5.5. *We have*

$$\begin{aligned} \left| \partial_x^\beta \partial_\xi^\alpha \int e^{-2i(z\eta - y\zeta)} b(X + Y) (e^{\epsilon\psi(X+Y) - \epsilon\psi(X+Z)} - 1) \hat{\chi}_2 dY dZ \right| \\ \leq C A^{|\alpha+\beta|} |\alpha + \beta|^{|\alpha+\beta|/\rho} e^{-c_1 \langle \xi \rangle_\gamma^\rho}. \end{aligned}$$

Repeating similar arguments we can prove

$$\begin{aligned} \left| \partial_x^\beta \partial_\xi^\alpha \int e^{-2i(\eta z - y\zeta)} b(X + Y) (e^{\epsilon\psi(X+Y) - \epsilon\psi(X+Z)} - 1) \hat{\chi}_i dY dZ \right| \\ \leq C A^{|\alpha+\beta|} |\alpha + \beta|^{|\alpha+\beta|/\rho} e^{-c_1 \langle \xi \rangle_\gamma^\rho} \end{aligned}$$

for $i = 3, 4$. We summarize what we have proved in

Proposition 5.1. *Let $b \in S_{(1)}(\omega^t \langle \xi \rangle_\gamma^m, \bar{g})$ then we have*

$$(be^{\epsilon\psi}) \# e^{-\epsilon\psi} = b + \omega^t \hat{b} + R$$

where $\hat{b} \in \sqrt{\epsilon} S_{(1)}(\langle \xi \rangle_\gamma^m, \bar{g})$ and $R \in S_{(1/\rho)}(e^{-c_1 \langle \xi \rangle_\gamma^\rho}, |dx|^2 + |d\xi|^2)$, that is

$$|\partial_x^\beta \partial_\xi^\alpha R| \leq C A^{|\alpha+\beta|} |\alpha + \beta|^{|\alpha+\beta|/\rho} e^{-c_1 \langle \xi \rangle_\gamma^\rho}.$$

5.3 Proo of Theorem 5.1

We start with the next lemma which is proved repeating similar arguments in the preceding subsection.

Lemma 5.6. *Let $a \in S_{(s)}(\langle \xi \rangle_\gamma^d, g_0)$ and $b \in S_{(1)}(\langle \xi \rangle_\gamma^h, \bar{g})$. Then we have*

$$(be^{\epsilon\psi}) \# a = \sum_{|\alpha+\beta| < N} \frac{(-1)^{|\beta|}}{(2i)^{|\alpha+\beta|} \alpha! \beta!} a_{(\alpha)}^{(\beta)} (be^{\epsilon\psi})_{(\beta)}^{(\alpha)} + b_N e^{\epsilon\psi} + R$$

where $b_N \in S_{(1)}(\langle \xi \rangle_\gamma^{d+h-\delta N}, \bar{g})$, $R \in S_{(1/\rho)}(e^{-c \langle \xi \rangle_\gamma^\rho}, |dx|^2 + |d\xi|^2)$. For $a \# (be^{\epsilon\psi})$ similar assertion holds, where $(-1)^{|\beta|}$ is replaced by $(-1)^{|\alpha|}$.

We can also prove

Lemma 5.7. *Let $R \in S_{(1/\rho)}(e^{-c\langle \xi \rangle_\gamma^\rho}, |dx|^2 + |d\xi|^2)$. Then we have*

$$R\#e^{\pm\epsilon\psi}, \quad e^{\pm\epsilon\psi}\#R \in S_{(1/\rho)}(e^{-c'\langle \xi \rangle_\gamma^\rho}, |dx|^2 + |d\xi|^2).$$

Corollary 5.4. *Let $R \in S_{(1/\rho)}(e^{-c\langle \xi \rangle_\gamma^\rho}, |dx|^2 + |d\xi|^2)$. Then for any $t \in \mathbb{R}$ we have*

$$e^{\pm\epsilon\psi}\#R\#e^{\mp\epsilon\psi} \in S(\langle \xi \rangle_\gamma^t, g_0).$$

Lemma 5.8. *Let $p \in S_{(s)}(\langle \xi \rangle_\gamma^d, g_0)$. Then one can write*

$$(e^{\epsilon\psi})\#p = \sum_{|\alpha+\beta|<N} \frac{(-1)^{|\beta|}}{i^{|\alpha+\beta|}\alpha!\beta!} p_{(\alpha)}^{(\beta)} \#(\omega_\beta^\alpha e^{\epsilon\psi}) + r_N e^{\epsilon\psi} + R$$

where $r_N \in S_{(1)}(\langle \xi \rangle_\gamma^{d-\delta N}, \bar{g})$, $R \in S_{(1/\rho)}(e^{-c\langle \xi \rangle_\gamma^\rho}, |dx|^2 + |d\xi|^2)$ and $\omega_\beta^\alpha = e^{-\epsilon\psi} \partial_x^\beta \partial_\xi^\alpha e^{\epsilon\psi}$.

Proof: We first examine

$$(5.5) \quad p_{(\alpha)}^{(\beta)} \omega_\beta^\alpha e^{\epsilon\psi} - \sum_{|\gamma+\delta|<N} \frac{(-1)^{|\gamma|}}{(2i)^{|\gamma+\delta|}\gamma!\delta!} p_{(\alpha+\delta)}^{(\beta+\gamma)} \#(\omega_{\beta+\gamma}^{\alpha+\delta} e^{\epsilon\psi}) = r_{N,|\alpha+\beta|} e^{\epsilon\psi} + R$$

with $r_{N,|\alpha+\beta|} \in S_{(1)}(\langle \xi \rangle_\gamma^{d-\delta N}, \bar{g})$. Indeed since $\partial_x^\nu \partial_\xi^\mu (\omega_\beta^\alpha e^{\epsilon\psi}) = \omega_{\beta+\nu}^{\alpha+\mu} e^{\epsilon\psi}$ thanks to Lemma 5.6 one can write

$$\begin{aligned} & \sum_{|\gamma+\delta|<N} \frac{(-1)^{|\gamma|}}{(2i)^{|\gamma+\delta|}\gamma!\delta!} p_{(\alpha+\delta)}^{(\beta+\gamma)} \#(\omega_{\beta+\gamma}^{\alpha+\delta} e^{\epsilon\psi}) \\ &= \sum_{|\gamma'+\delta'|<2N} \frac{(-1)^{|\gamma'|}}{(2i)^{|\gamma'+\delta'|}\gamma'!\delta'!} \left(\sum \binom{\gamma'}{\mu} \binom{\delta'}{\nu} (-1)^{|\mu+\nu|} \right) p_{(\alpha+\delta')}^{(\beta+\gamma')} \omega_{\beta+\gamma'}^{\alpha+\delta'} e^{\epsilon\psi} \\ &+ r_{N,|\alpha+\beta|} e^{\epsilon\psi} + R \end{aligned}$$

where $\sum \binom{\gamma'}{\mu} \binom{\delta'}{\nu} (-1)^{|\mu+\nu|} = 0$ if $|\gamma'| + |\delta'| > 0$ so that the right-hand side is

$$p_{(\alpha)}^{(\beta)} \omega_\beta^\alpha e^{\epsilon\psi} + r_{N,|\alpha+\beta|} e^{\epsilon\psi} + R, \quad r_{N,|\alpha+\beta|} \in S_{(1)}(\langle \xi \rangle_\gamma^{d-\delta N}, \bar{g})$$

which proves (5.5). Now insert the expression of $p_{(\alpha)}^{(\beta)} \omega_\beta^\alpha e^{\epsilon\psi}$ in (5.5) into

$$(e^{\epsilon\psi})\#p = \sum_{|\alpha+\beta|<N} \frac{(-1)^{|\beta|}}{(2i)^{|\alpha+\beta|}\alpha!\beta!} p_{(\alpha)}^{(\beta)} \omega_\beta^\alpha e^{\epsilon\psi} + r_N e^{\epsilon\psi} + R$$

which follows from Lemma 5.6 to get

$$\sum_{|\alpha' + \beta'| < 2N} \frac{(-1)^{|\beta'|}}{(2i)^{|\alpha' + \beta'|} \alpha'! \beta'!} \left(\sum \binom{\alpha'}{\delta} \binom{\beta'}{\gamma} \right) p_{(\alpha')}^{(\beta')} \# (\omega_{\beta'}^{\alpha'} e^{\epsilon\psi}) + \tilde{r}_N e^{\epsilon\psi} + R$$

where $\tilde{r}_N \in S_{(1)}(\langle \xi \rangle_{\gamma}^{d-\delta N}, \bar{g})$. Here we note $\sum \binom{\alpha'}{\delta} \binom{\beta'}{\gamma} = 2^{|\alpha' + \beta'|}$. It is clear that $p_{(\alpha')}^{(\beta')} \# (\omega_{\beta'}^{\alpha'} e^{\epsilon\psi}) = r' e^{\epsilon\psi} + R$ with $r' \in S_{(1)}(\langle \xi \rangle_{\gamma}^{d-\delta N}, \bar{g})$ for $|\alpha' + \beta'| \geq N$ and hence we get the assertion. \square

Proof of Theorem 5.1: From Lemma 5.8 we see

$$(e^{\epsilon\psi}) \# p \# e^{-\epsilon\psi} = \sum_{|\alpha + \beta| \leq m} \frac{(-1)^{|\beta|}}{i^{|\alpha + \beta|} \alpha! \beta!} p_{(\alpha)}^{(\beta)} \# ((\omega_{\beta}^{\alpha} e^{\epsilon\psi}) \# e^{-\epsilon\psi}) + (r_m e^{\epsilon\psi} + R) \# e^{-\epsilon\psi}$$

where $(r_m e^{\epsilon\psi} + R) \# e^{-\epsilon\psi} \in S(\langle \xi \rangle_{\gamma}^{m-\delta(m+1)}, \bar{g})$ which follows from Propositions 5.1 and Lemma 5.7. Therefore Propositions 5.1 together with Corollary 5.3 gives

$$(e^{\epsilon\psi}) \# p \# e^{-\epsilon\psi} = \sum_{|\alpha + \beta| \leq m} \frac{(-1)^{|\beta|}}{i^{|\alpha + \beta|} \alpha! \beta!} p_{(\alpha)}^{(\beta)} \# (\omega_{\beta}^{\alpha} + \bar{\omega}_{\beta}^{\alpha}) + R$$

where $\bar{\omega}_{\beta}^{\alpha} \in \epsilon^{|\alpha + \beta| + 1/2} S_{(1)}(\langle \xi \rangle_{\gamma}^{\rho|\beta| - \delta|\alpha|}, \bar{g})$, $R \in S(\langle \xi \rangle_{\gamma}^{m-\delta(m+1)}, \bar{g})$ and

$$\bar{\omega}_{\beta}^{\alpha} \in \epsilon^{3/2} S_{(1)}(\omega^{-1} \langle \xi \rangle_{\gamma}^{\kappa - |\alpha|}, \bar{g}), \quad |\alpha + \beta| = 1.$$

Since $e^{\epsilon\psi} \# e^{-\epsilon\psi} = 1 - r_1$, $r_1 \in \sqrt{\epsilon} S(1, \bar{g})$ by Proposition 5.1 there exists $K = 1 + r$, $r \in \sqrt{\epsilon} S(1, \bar{g})$ such that $e^{\epsilon\psi} \# e^{-\epsilon\psi} \# K = 1$ if $0 < \epsilon \leq \epsilon_0$ is small ([1, Theorem 3.2] and [15, Theorem 2.6.27] for example). Thus we have

$$(e^{\epsilon\psi}) \# p \# e^{-\epsilon\psi} \# K = p + \sum_{1 \leq |\alpha + \beta| \leq m} \frac{(-1)^{|\beta|}}{i^{|\alpha + \beta|} \alpha! \beta!} p_{(\alpha)}^{(\beta)} \# (\omega_{\beta}^{\alpha} + \bar{\omega}_{\beta}^{\alpha}) \# K + R.$$

On the other hand it is clear $(\omega_{\beta}^{\alpha} + \bar{\omega}_{\beta}^{\alpha}) \# (1 + r) = \omega_{\beta}^{\alpha} + \bar{\omega}_{\beta}^{\alpha}$ with $\bar{\omega}_{\beta}^{\alpha} \in \epsilon^{|\alpha + \beta| + 1/2} S(\langle \xi \rangle_{\gamma}^{\rho|\beta| - \delta|\alpha|}, \bar{g})$. Since $\omega_{\beta}^{\alpha} \in \epsilon S(\omega^{-1} \langle \xi \rangle_{\gamma}^{\kappa - |\alpha|}, \bar{g})$ for $|\alpha + \beta| = 1$ it is also clear that $\bar{\omega}_{\beta}^{\alpha} \in \epsilon^{3/2} S(\omega^{-1} \langle \xi \rangle_{\gamma}^{\kappa - |\alpha|}, \bar{g})$ for $|\alpha + \beta| = 1$. Note

$$\begin{aligned} p_{(\alpha)}^{(\beta)} \# (\omega_{\beta}^{\alpha} + \bar{\omega}_{\beta}^{\alpha}) - \sum_{|\mu + \nu| \leq m - |\alpha + \beta|} \frac{(-1)^{|\nu|}}{i^{|\mu + \nu|} \mu! \nu!} p_{(\alpha + \nu)}^{(\beta + \mu)} (\omega_{\beta}^{\alpha} + \bar{\omega}_{\beta}^{\alpha})_{(\mu)}^{(\nu)} \\ \in S(\langle \xi \rangle_{\gamma}^{m - \delta|\alpha + \beta| - \rho(m+1 - |\alpha + \beta|)}, \bar{g}) \subset S(\langle \xi \rangle_{\gamma}^{m - \delta(m+1)}, \bar{g}) \end{aligned}$$

and $(\omega_\beta^\alpha + \bar{\omega}_\beta^\alpha)_{(\mu)}^{(\nu)} \in \gamma^{-\kappa|\mu+\nu|} S_{\langle 1 \rangle}(\langle \xi \rangle_\gamma^{\rho|\alpha+\mu|-\delta|\beta+\nu|}, \bar{g})$ which is contained in $\epsilon^{|\alpha+\beta|+|\mu+\nu|+1/2} S_{\langle 1 \rangle}(\langle \xi \rangle_\gamma^{\rho|\alpha+\mu|-\delta|\beta+\nu|}, \bar{g})$ if $|\mu+\nu| \geq 1$, $\gamma \geq \gamma_0(\epsilon)$ so that

$$(e^{\epsilon\psi}) \# p \# e^{-\epsilon\psi} \# K = p + \sum_{1 \leq |\alpha+\beta| \leq m} \frac{(-1)^{|\beta|}}{i^{|\alpha+\beta|} \alpha! \beta!} p_{(\alpha)}^{(\beta)} (\omega_\beta^\alpha + \hat{\omega}_\beta^\alpha) + R$$

where $\hat{\omega}_\beta^\alpha \in \epsilon^{|\alpha+\beta|+1/2} S(\langle \xi \rangle_\gamma^{\rho|\beta|-\delta|\alpha|}, \bar{g})$ and $\hat{\omega}_\beta^\alpha \in \epsilon^{3/2} S(\omega^{-1} \langle \xi \rangle_\gamma^{\kappa-|\alpha|}, \bar{g})$ for $|\alpha+\beta| = 1$. Now check ω_β^α . For $|\alpha+\beta| = 1$ we have $\omega_\beta^\alpha = \epsilon(-i\nabla_\xi \psi)^\alpha (i\nabla_x \psi)^\beta$. Let $|\alpha+\beta| \geq 2$ then ω_β^α is a linear combination of terms $(\epsilon\psi)_{(\beta_1)}^{(\alpha_1)} \cdots (\epsilon\psi)_{(\beta_s)}^{(\alpha_s)}$ with $\alpha_1 + \cdots + \alpha_s = \alpha$, $\beta_1 + \cdots + \beta_s = \beta$, $|\alpha_i + \beta_i| \geq 1$. If $|\alpha_i + \beta_i| = 1$ for all i it is clear $\omega_\beta^\alpha = \epsilon^{|\alpha+\beta|} (-i\nabla_\xi \psi)^\alpha (i\nabla_x \psi)^\beta$. If $|\alpha_j + \beta_j| \geq 2$ for some j so that $s \leq |\alpha+\beta| - 2$ then one has

$$(\epsilon\psi)_{(\beta_1)}^{(\alpha_1)} \cdots (\epsilon\psi)_{(\beta_s)}^{(\alpha_s)} \in S(\langle \xi \rangle_\gamma^{-\kappa+\rho|\beta|-\delta|\alpha|}, \bar{g}) \subset \gamma^{-\kappa} S(\langle \xi \rangle_\gamma^{\rho|\alpha|-\delta|\beta|}, \bar{g}).$$

Since we can assume $\gamma^{-\kappa} \leq \epsilon^{|\alpha+\beta|+1/2}$ for $\gamma \geq \gamma_0(\epsilon)$ we get the assertion. \square

6 Energy estimates

To obtain energy estimates we follow [13] where the main point is to derive microlocal energy estimates. We sketch how to get microlocal energy estimates. Let us denote

$$P_\psi = \text{Op}(e^{\epsilon\psi}) P \text{Op}(e^{-\epsilon\psi}) \text{Op}(K)$$

of which principal symbol is given by $p_\psi = e^{\epsilon\psi} \# p \# e^{-\epsilon\psi} \# K$. In this section we say $a(x, \xi; \gamma, \epsilon) \in \tilde{S}(W, \bar{g})$ if $a \in S(W, \bar{g})$ for each fixed $0 < \epsilon \ll 1$. Let $a \in \tilde{S}(W, \bar{g})$ and let $N \in \mathbb{N}$ be given. Then with a fixed small $0 < 2\tau < \rho - \delta$ we have

$$|\partial_x^\beta \partial_\xi^\alpha a| \leq C_{\alpha\beta}(\epsilon) W \langle \xi \rangle_\gamma^{-\rho|\alpha|+\delta|\beta|} \leq C_{\alpha\beta} \gamma^{-2\tau|\alpha+\beta|} W \langle \xi \rangle_\gamma^{-(\rho-\tau)|\alpha|+(\delta+\tau)|\beta|}$$

where one can assume that $C_{\alpha\beta}(\epsilon) \gamma^{-2\tau|\alpha+\beta|}$ are arbitrarily small for $1 \leq |\alpha+\beta| \leq N$ taking γ large.

6.1 Symbol of P_ψ

Define $h_j(x, \xi)$ by

$$h_j(x, \xi) = \sum_{1 \leq \ell_1 < \ell_2 < \cdots < \ell_j \leq m} |q_{\ell_1}|^2 \cdots |q_{\ell_j}|^2, \quad q_j = \xi_0 - \lambda_j(x, \xi').$$

Lemma 6.1. *There exists $c > 0$ such that*

$$h_{m-k}(x, \xi - i\epsilon\omega^{-1}\langle \xi \rangle_{\gamma}^{\kappa}\theta) \geq c(\epsilon\omega)^{2(j-k)}\langle \xi \rangle_{\gamma}^{2(j-k)}h_{m-j}(x, \xi - i\epsilon\omega^{-1}\langle \xi \rangle_{\gamma}^{\kappa}\theta)$$

for $j = k, \dots, m$ where $h_0 = 1$ and $1 \leq k \leq m$.

Proof: We show the case $k = 1$. By definition $h_{m-1}(x, \xi - i\epsilon\omega^{-1}\langle \xi \rangle_{\gamma}^{\kappa}\theta)$ is bounded from below by

$$2^{-1}(|q_i(x, \xi)|^2 + |q_j(x, \xi)|^2 + \epsilon^2\omega^{-2}\langle \xi \rangle_{\gamma}^{2\kappa}) \prod_{k \neq i, j} |q_k(x, \xi - i\epsilon\omega^{-1}\langle \xi \rangle_{\gamma}^{\kappa}\theta)|^2.$$

From Lemma 3.2 we have $|q_i(x, \xi)|^2 + |q_j(x, \xi)|^2 \geq c|b'(x, \xi)|^2$. Since

$$\begin{aligned} & |b'(x, \xi)|^2 + \epsilon^2\omega^{-2}\langle \xi \rangle_{\gamma}^{2\kappa} \\ &= \epsilon^2\omega^{-2}\langle \xi \rangle_{\gamma}^2(\epsilon^{-2}|b'(x, \xi)|^2\omega^2\langle \xi \rangle_{\gamma}^{-2} + \langle \xi \rangle_{\gamma}^{-4\delta}) \geq c\epsilon^2\omega^2\langle \xi \rangle_{\gamma}^2 \end{aligned}$$

with some $c > 0$ because $C|b'(x, \xi)|^2\langle \xi \rangle_{\gamma}^{-2} \geq \phi^2$ and $\omega^2 \geq \phi^2$ then it is clear that $h_{m-1}(x, \xi - i\epsilon\omega^{-1}\langle \xi \rangle_{\gamma}^{\kappa}\theta)$ is bounded from below by

$$c\epsilon^2\omega^2\langle \xi \rangle_{\gamma}^2 \prod_{k \neq i, j} |q_k(x, \xi - i\epsilon\omega^{-1}\langle \xi \rangle_{\gamma}^{\kappa}\theta)|^2.$$

Summing up over all pair i, j ($i \neq j$) we get the assertion for the case $j = 2$. Continuing this argument one can prove the case $j \geq 3$. \square

Let us put

$$h(x, \xi) = h_{m-1}(x, \xi - i\epsilon\omega^{-1}\langle \xi \rangle_{\gamma}^{\kappa}\theta)^{1/2}.$$

Lemma 6.2. *There exists $C > 0$ such that we have*

$$\begin{cases} |p_{(\beta)}^{(\alpha)}| \leq C(\epsilon\omega)^{1-|\alpha+\beta|}\langle \xi \rangle_{\gamma}^{1-|\alpha|}h, & 1 \leq |\alpha+\beta| \leq m, \\ |p_{(\beta)}^{(\alpha)}| \leq C(\epsilon\omega)^{2-|\alpha+\beta|}\langle \xi \rangle_{\gamma}^{2-|\alpha|}h^2, & 2 \leq |\alpha+\beta| \leq m. \end{cases}$$

Proof: From [4, Proposition 3] one has

$$|p_{(\beta)}^{(\alpha)}(x, \xi)| \leq Ch_{m-|\alpha+\beta|}(x, \xi)^{1/2}|\xi|^{|\beta|}$$

for $|\alpha+\beta| \leq m$ which is bounded by $Ch_{m-|\alpha+\beta|}(x, \xi - i\epsilon\omega^{-1}\langle \xi \rangle_{\gamma}^{\kappa}\theta)^{1/2}|\xi|^{|\beta|}$ clearly. On the other hand it follows from Lemma 6.1

$$Ch(x, \xi) \geq (\epsilon\omega)^{|\alpha+\beta|-1}\langle \xi \rangle_{\gamma}^{|\alpha+\beta|-1}h_{m-|\alpha+\beta|}(x, \xi - i\epsilon\omega^{-1}\langle \xi \rangle_{\gamma}^{\kappa}\theta)^{1/2}$$

for $1 \leq |\alpha+\beta| \leq m$ which proves the assertion. The proof of the second assertion is similar. \square

Lemma 6.3. Assume that $c_\beta^\alpha \in S(\langle \xi \rangle_\gamma^{\rho|\beta|-\delta|\alpha|}, \bar{g})$ and $c_\beta^\alpha \in S(\omega^{-1} \langle \xi \rangle_\gamma^{\kappa-|\alpha|}, \bar{g})$ for $|\alpha + \beta| = 1$. Then for $1 \leq |\alpha + \beta| \leq m$ we have $p_{(\beta)}^{(\alpha)} c_\alpha^\beta \in \tilde{S}(\omega^{-1} \langle \xi \rangle_\gamma^\kappa h, \bar{g})$ and $\epsilon^{|\alpha+\beta|} |p_{(\beta)}^{(\alpha)} c_\alpha^\beta| \leq C \epsilon \omega^{-1} \langle \xi \rangle_\gamma^\kappa h$ with $C > 0$ independent of ϵ .

Proof: Let $2 \leq |\alpha + \beta| \leq m$ then since $1 = \kappa + 2\delta$ we see by Lemma 6.2

$$\begin{aligned} \epsilon^{|\alpha+\beta|} |p_{(\beta)}^{(\alpha)} c_\alpha^\beta| &\leq C \epsilon \omega^{1-|\alpha+\beta|} \langle \xi \rangle_\gamma^{1-|\alpha|+\rho|\alpha|-\delta|\beta|} h \\ &\leq C \epsilon \omega^{-1} \langle \xi \rangle_\gamma^\kappa (\omega^{-1} \langle \xi \rangle_\gamma^{-\delta})^{|\alpha+\beta|-2} h \leq C \epsilon \omega^{-1} \langle \xi \rangle_\gamma^\kappa h. \end{aligned}$$

When $|\alpha + \beta| = 1$ noting $c_\alpha^\beta \in S(\omega^{-1} \langle \xi \rangle_\gamma^{\kappa-|\beta|}, \bar{g})$ we get the same assertion. We next estimate $\sum p_{(\beta+\nu')}^{(\alpha+\mu')}(c_\alpha^\beta)_{(\nu'')}^{(\mu'')}$. If $|\alpha + \mu' + \beta + \nu'| \geq m$ we have

$$\begin{aligned} |p_{(\beta+\nu')}^{(\alpha+\mu')} \omega_{\alpha(\nu'')}^{\beta(\mu'')}| &\leq \langle \xi \rangle_\gamma^{m-|\alpha+\mu'|} \langle \xi \rangle_\gamma^{\rho|\alpha|-\delta|\beta|} \langle \xi \rangle_\gamma^{-\rho|\mu''|+\delta|\nu''|} \\ &\leq C_\epsilon \omega^{-(m-1)} h \langle \xi \rangle_\gamma^{1-|\alpha+\mu'|} \langle \xi \rangle_\gamma^{\rho|\alpha|-\delta|\beta|+\rho|\mu'|-\delta|\nu'|} \langle \xi \rangle_\gamma^{-\rho|\mu|+\delta|\nu|} \\ &\leq C_\epsilon \omega^{-1} \langle \xi \rangle_\gamma^\kappa (\omega^{-(m-2)} \langle \xi \rangle_\gamma^{-\delta(|\alpha+\mu'+\beta+\nu'|-2)}) \langle \xi \rangle_\gamma^{-\rho|\mu|+\delta|\nu|} h \end{aligned}$$

where the right-hand side is bounded by $C_\epsilon \omega^{-1} \langle \xi \rangle_\gamma^{\kappa-\rho|\mu|+\delta|\nu|} h$. We turn to the case $|\alpha + \mu' + \beta + \nu'| \leq m$. From Lemma 6.2 it follows

$$\begin{aligned} |p_{(\beta+\nu')}^{(\alpha+\mu')}| &\leq C_\epsilon \omega^{1-|\alpha+\mu'+\beta+\nu'|} \langle \xi \rangle_\gamma^{1-|\alpha+\mu'|} h \\ &\leq C_\epsilon (\omega^{-1} \langle \xi \rangle_\gamma^{-\delta})^{|\mu'+\nu'|} \omega^{1-|\alpha+\beta|} \langle \xi \rangle_\gamma^{1-|\alpha|} h \langle \xi \rangle_\gamma^{-\rho|\mu'|+\delta|\nu'|} \end{aligned}$$

therefor for $|\alpha + \beta| \geq 2$ we see easily

$$\begin{aligned} |p_{(\beta+\nu')}^{(\alpha+\mu')} c_{\alpha(\nu'')}^{\beta(\mu'')}| &\leq C_\epsilon (\omega^{-1} \langle \xi \rangle_\gamma^{-\delta})^{|\alpha+\beta|-2} \omega^{-1} \langle \xi \rangle_\gamma^\kappa h \langle \xi \rangle_\gamma^{-\rho|\mu|+\delta|\nu|} \\ &\leq C_\epsilon \omega^{-1} \langle \xi \rangle_\gamma^\kappa h \langle \xi \rangle_\gamma^{-\rho|\mu|+\delta|\nu|} \end{aligned}$$

which also holds for $|\alpha + \beta| = 1$ because $c_\alpha^\beta \in S(\omega^{-1} \langle \xi \rangle_\gamma^{\kappa-|\beta|}, \bar{g})$. \square

6.2 Definition of $Q(z)$ which separates $P_\psi(z)$

We follow the arguments in [13]. Let us define $\tilde{p}(x + iy, \xi + i\eta)$ by

$$\tilde{p}(x + iy, \xi + i\eta) = \sum_{|\alpha+\beta| \leq m} \frac{1}{\alpha! \beta!} \partial_x^\alpha \partial_\xi^\beta p(x, \xi) (iy)^\alpha (i\eta)^\beta.$$

Then p_ψ given by Theorem 5.1 is expressed as $\tilde{p}(z - i\epsilon H_\psi)$, which one can also write as

$$\tilde{p}(z - i\epsilon H_\psi) = \sum_{j=0}^m \left(i \frac{\partial}{\partial t}\right)^j p(z - \epsilon t H_\psi) / j! \Big|_{t=0}.$$

Using this expression we define $Q(z)$ which separates $\tilde{p}(z - i\epsilon H_\psi)$ by

$$Q(z) = \epsilon^{-1} |\tilde{H}_\psi|^{-1} \left(\frac{\partial}{\partial t} \right) \sum_{j=0}^m \left(i \frac{\partial}{\partial t} \right)^j p(z - \epsilon t H_\psi) / j! \Big|_{t=0}$$

where $\tilde{H}_\psi = (\langle \xi \rangle_\gamma \nabla_\xi \psi, -\nabla_x \psi)$. By the homogeneity it is clear that

$$\tilde{p}(z - i\epsilon H_\psi) = \langle \xi \rangle_\gamma^m \tilde{p}(\tilde{z} - i\lambda(z) \tilde{H}_\psi / |\tilde{H}_\psi|), \quad \lambda(x, \xi) = \epsilon |\tilde{H}_\psi| \langle \xi \rangle_\gamma^{-1}$$

where $\tilde{z} = (x, \xi \langle \xi \rangle_\gamma^{-1})$. It is not difficult to check $\tilde{p}(z - i\epsilon H_\psi) \in S(\langle \xi \rangle_\gamma^m, \bar{g})$ and $Q \in S(\langle \xi \rangle_\gamma^{m-1}, \bar{g})$. We study $\tilde{p}(z - i\epsilon H_\psi)$ and $Q(z)$ in a conic neighborhood of ρ . We first recall

Proposition 6.1 ([14, Lemma 5.8]). *Let ρ be a characteristic of p of order m and let $K \subset \Gamma_\rho$ be a compact set. Then one can find a conic neighborhood V of ρ and positive $C > 0$ such that for any $(x, \xi) \in V$, $\zeta \in K$ and small $s \in \mathbb{R}$ one can write*

$$p(z - s\zeta) = e(z, \zeta, s) \prod_{j=1}^m (s - \mu_j(z, \zeta))$$

where $\mu_j(z, \zeta)$ are real valued and $e(z, \zeta, s) \neq 0$ for $(z, \zeta, s) \in V \times K \times \{|s| < s_0\}$. Moreover there exists $C > 0$ such that we have

$$(6.1) \quad |\mu_j(z, (0, \theta))|/C \leq |\mu_j(z, \zeta)| \leq C |\mu_j(z, (0, \theta))|, \quad j = 1, 2, \dots, m$$

for any $(x, \xi) \in V$, $\zeta \in K$.

Writing $\prod_{j=1}^m (t - \mu_j) = \sum_{\ell=0}^m p_\ell t^\ell$ we see that $\tilde{p}(z - is\zeta)$ is written

$$\sum_{j=0}^m \frac{1}{j!} \left(is \frac{\partial}{\partial t} \right)^j \left(e \prod_{j=1}^m (t - \mu_j) \right) \Big|_{t=0} = \sum_{\ell=0}^m \sum_{k=0}^{m-\ell} \frac{1}{k!} \left(is \frac{\partial}{\partial t} \right)^k e \Big|_{t=0} p_\ell (is)^\ell$$

which is equal to

$$\begin{aligned} & \sum_{\ell=0}^m \left(\sum_{k=0}^m \frac{1}{k!} \left(is \frac{\partial}{\partial t} \right)^k e \Big|_{t=0} - \sum_{m-\ell+1 \leq k \leq m} \frac{1}{k!} \left(is \frac{\partial}{\partial t} \right)^k e \Big|_{t=0} \right) p_\ell (is)^\ell \\ &= \sum_{k=0}^m \frac{1}{k!} \left(is \frac{\partial}{\partial t} \right)^k e \Big|_{t=0} \prod_{j=1}^m (is - \mu_j) + O(s^{m+1}) \end{aligned}$$

which proves

$$(6.2) \quad \tilde{p}(z - is\zeta) = e_0(z, \zeta, s) \prod_{k=1}^m (is - \mu_j(z, \zeta)) + O(s^{m+1}).$$

Note that $e_0(z, \zeta, s) = \sum_{k=0}^m (is \partial / \partial t)^k e(z, \zeta, t) / k!|_{t=0}$ and hence we have $e_0(z, \zeta, 0) = e(z, \zeta, 0) \neq 0$.

Lemma 6.4. *There exist a conic neighborhood U of ρ and a compact convex set $K \subset \Gamma_\rho$ such that $\tilde{H}_\psi / |\tilde{H}_\psi| \in K$ for $(x, \xi) \in U$, $\gamma \geq \gamma_0$.*

Proof: From $\psi = \langle \xi \rangle_\gamma^\kappa \log(\phi + \omega)$ it is easy to see

$$\begin{cases} \nabla_x \psi = \omega^{-1} \langle \xi \rangle_\gamma^\kappa \nabla_x \phi, \\ \nabla_\xi \psi = \omega^{-1} \langle \xi \rangle_\gamma^\kappa \nabla_\xi \phi + O(\langle \xi \rangle_\gamma^{\kappa-1}) \log(\phi + \omega) + O(\langle \xi \rangle_\gamma^{\kappa-1}). \end{cases}$$

Therefore one has

$$|\tilde{H}_\psi|^2 \tilde{H}_\psi = \omega^{-1} \langle \xi \rangle_\gamma^\kappa (\tilde{H}_\phi + (O(1)\omega \log(\phi + \omega), 0)).$$

Since $|\phi + \omega| \leq 2\omega$ we can assume $\omega \log(\phi + \omega)$ is enough small taking U small. In particular we have $\omega^{-1} \langle \xi \rangle_\gamma^\kappa / C \leq |\tilde{H}_\psi| \leq C\omega^{-1} \langle \xi \rangle_\gamma^\kappa$. Then noting $\tilde{H}_\phi(\rho) / |\tilde{H}_\phi(\rho)| \in \Gamma_\rho$ which follows from (4.3) we get the assertion. \square

We rewrite $Q(z)$ according to (6.2).

Lemma 6.5. *Let $\tilde{\omega} = \tilde{H}_\psi / |\tilde{H}_\psi|$. Then we have*

$$\begin{aligned} Q(z) &= \langle \xi \rangle_\gamma^{m-1} \{ -i \partial e_0(\tilde{z}, \tilde{\omega}, \lambda) / \partial \lambda \prod_{j=1}^m (i\lambda - \mu_j(\tilde{z}, \tilde{\omega})) \\ &\quad + e_0(\tilde{z}, \tilde{\omega}, \lambda) \sum_{j=1}^m \prod_{k=1, k \neq j}^m (i\lambda - \mu_k(\tilde{z}, \tilde{\omega})) + O(\lambda^m) \}. \end{aligned}$$

Proof: Noting $\lambda(z) |\tilde{H}_\psi|^{-1} \langle \xi \rangle_\gamma = \epsilon$ one can write

$$\begin{aligned} Q(z) &= \epsilon^{-1} |\tilde{H}_\psi|^{-1} \langle \xi \rangle_\gamma^m \sum_{j=0}^m \left(\frac{\partial}{\partial t} \right) \left(i \frac{\partial}{\partial t} \right)^j p(\tilde{z} - t\lambda(z)\tilde{\omega}) / j!|_{t=0} \\ &= \langle \xi \rangle_\gamma^{m-1} \sum_{j=0}^m \lambda(z)^j \left(\frac{\partial}{\partial t} \right) \left(i \frac{\partial}{\partial t} \right)^j p(\tilde{z} - t\tilde{\omega}) / j!|_{t=0} \end{aligned}$$

which is equal to

$$\begin{aligned} (6.3) \quad & \langle \xi \rangle_\gamma^{m-1} \frac{\partial}{\partial s} \frac{1}{i} \sum_{j=0}^m \frac{1}{(j+1)!} \left(is \frac{\partial}{\partial t} \right)^{j+1} p(\tilde{z} - t\tilde{\omega})|_{t=0, s=\lambda(z)} \\ &= \frac{1}{i} \langle \xi \rangle_\gamma^{m-1} \frac{\partial}{\partial s} \{ \tilde{p}(\tilde{z} - is\tilde{\omega}) - p(\tilde{z}) + O(s^{m+1}) \}_{s=\lambda(z)} \\ &= \frac{1}{i} \langle \xi \rangle_\gamma^{m-1} \left\{ \frac{\partial}{\partial s} \tilde{p}(\tilde{z} - is\tilde{\omega})|_{s=\lambda(z)} + O(\lambda^m) \right\}. \end{aligned}$$

From (6.2) the right-hand side of (6.3) turns to be

$$(6.4) \quad \frac{1}{i} \langle \xi \rangle_\gamma^{m-1} \left\{ \frac{\partial}{\partial s} (e_0(\tilde{z}, \tilde{\omega}, s) \prod_{j=1}^m (is - \mu_j(\tilde{z}, \tilde{\omega})) + O(s^{m+1})) \right\}_{s=\lambda}$$

modulo $O(\lambda^m) \langle \xi \rangle_\gamma^{m-1}$ which proves the assertion. \square

6.3 Microlocal energy estimates

To derive microlocal energy estimates we study $\text{Im}(P_\psi \chi u, Q \chi u)$ where χ is a cutoff symbol supported in a conic neighborhood of ρ . Thus we are led to consider $\text{Im}(P_\psi \bar{Q})$ in a conic neighborhood of ρ . Recall that $P_\psi = p_\psi + \sum_{j=0}^{m-1} (P_j)_\psi$ where $P_j(x, D)$ is the homogeneous part of degree j of P and $(P_j)_\psi \in S(\langle \xi \rangle_\gamma^j, \bar{g})$ by Theorem 5.1. Take any small $0 < \epsilon^* \ll 1$ and we fix ϵ^* and put

$$\delta = (1 - \epsilon^*)/m, \quad \rho = (m - 1 + \epsilon^*)/m, \quad \kappa = \rho - \delta.$$

Lemma 6.6. *Let $S_0(z) = \text{Im}(\tilde{p}(z - i\epsilon H_\psi) \overline{Q(z)})$. Then one can find a conic neighborhood V of ρ and $C > 0$ such that we have*

$$\epsilon \omega^{-1} \langle \xi \rangle_\gamma^\kappa h^2(z)/C \leq S_0(z) \leq C \epsilon \omega^{-1} \langle \xi \rangle_\gamma^\kappa h^2(z).$$

Proof: Write $e_0(\tilde{z}, \tilde{\omega}, \lambda) = e(\tilde{z}, \tilde{\omega}, 0) + i\lambda(\partial e/\partial \lambda)(\tilde{z}, \tilde{\omega}, 0) + O(\lambda^2)$ then it is clear $|e_0|^2 = |e(\tilde{z}, \tilde{\omega}, 0)|^2 + O(\lambda^2)$ so that $\text{Re}(\partial \bar{e}_0/\partial \lambda) e_0 = 2^{-1} \partial |e_0|^2 / \partial \lambda = O(\lambda)$. Thus from Lemma 6.5 and (6.2) it follows that

$$\begin{aligned} \text{Im}(\tilde{p}(z - i\epsilon H_\psi) \overline{Q(z)}) &= \langle \xi \rangle_\gamma^{2m-1} |e_0|^2 \lambda \\ &\times \left\{ \sum_{j=1}^m \prod_{k=1, k \neq j}^m (\lambda^2 + \mu_k^2) (1 + O(\lambda + \sum_{j=1}^m |\mu_j|)) \right\}. \end{aligned}$$

Since $\mu_j(\rho, \tilde{\omega}) = 0$, $j = 1, 2, \dots, m$ one obtains

$$S_0(z) \approx \langle \xi \rangle_\gamma^{2m-1} \lambda \sum_{j=1}^m \prod_{k=1, k \neq j}^m (\lambda^2 + \mu_k(\tilde{z}, \tilde{\omega})^2).$$

On the other hand noting $\lambda(z) \approx \epsilon \omega^{-1} \langle \xi \rangle_\gamma^{\kappa-1}$ and

$$\begin{aligned} &h_{m-1}(x, \xi - i\epsilon \omega^{-1} \langle \xi \rangle_\gamma^\kappa \theta) \\ &\approx \langle \xi \rangle_\gamma^{2m-2} \sum_{j=1}^m \prod_{k=1, k \neq j}^m (\epsilon^2 \omega^{-2} \langle \xi \rangle_\gamma^{2\kappa-2} + \mu_k(\tilde{z}, (0, \theta))^2) \end{aligned}$$

we conclude the assertion from Lemma 6.4 and Proposition 6.1. \square

Lemma 6.7. *We have $Q \in \tilde{S}(h, \bar{g})$ and $S_0^{\pm 1} \in \tilde{S}((\omega^{-1}\langle \xi \rangle_\gamma^\kappa h^2)^{\pm 1}, \bar{g})$ in U for $\gamma \geq \gamma_0(\epsilon)$. Moreover $|Q| \leq Ch$ with $C > 0$ independent of ϵ .*

Proof: From the definition one can see easily that Q is a sum of terms, up to constant factor;

$$(6.5) \quad \epsilon^{|\alpha+\beta|-1} p_{(\beta)}^{(\alpha)}(z) (\nabla_\xi \psi)^\beta (\nabla_x \psi)^\alpha / ((\langle \xi \rangle_\gamma^2 |\nabla_\xi \psi|^2 + |\nabla_x \psi|^2)^{1/2})$$

with $1 \leq |\alpha + \beta| \leq m + 1$. We also note that $\text{Im } Q$ is a sum of such terms (6.5) with $2 \leq |\alpha + \beta| \leq m + 1$. From Lemma 4.3 it follows

$$\begin{cases} \nabla_\xi \psi \in S(\omega^{-1}\langle \xi \rangle_\gamma^{\kappa-1}, \bar{g}), \\ \nabla_x \psi \in S(\omega^{-1}\langle \xi \rangle_\gamma^\kappa, \bar{g}) \end{cases}$$

in U and hence $((\langle \xi \rangle_\gamma^2 |\nabla_\xi \psi|^2 + |\nabla_x \psi|^2)^{-1/2}) \in S((\omega^{-1}\langle \xi \rangle_\gamma^\kappa)^{-1}, \bar{g})$ then we have

$$\begin{aligned} V_\alpha^\beta &= (\nabla_\xi \psi)^\beta (\nabla_x \psi)^\alpha ((\langle \xi \rangle_\gamma^2 |\nabla_\xi \psi|^2 + |\nabla_x \psi|^2)^{-1/2}) \\ &\in S(\omega \langle \xi \rangle_\gamma^{-\kappa} \langle \xi \rangle_\gamma^{\rho|\alpha| - \delta|\beta|}, \bar{g}). \end{aligned}$$

Noting $\omega^{-2}\langle \xi \rangle_\gamma^{\kappa-1} \leq 1$ it suffices to repeat the proof of Lemma 6.3 to conclude $p_{(\beta)}^{(\alpha)} V_\alpha^\beta \in \tilde{S}(h, \bar{g})$ and $\epsilon^{|\alpha+\beta|-1} |p_{(\beta)}^{(\alpha)} V_\alpha^\beta| \leq Ch$ with C independent of ϵ for $1 \leq |\alpha + \beta| \leq m + 1$. From Lemma 6.3 it follows that $\tilde{p}(z - i\epsilon H_\psi) - p(z) \in \tilde{S}(\omega^{-1}\langle \xi \rangle_\gamma^\kappa h, \bar{g})$. Since from Lemma 6.2 one can check that $p(z)\text{Im } Q \in \tilde{S}(\omega^{-1}\langle \xi \rangle_\gamma^\kappa h^2, \bar{g})$ we get the assertion for S_0 . The assertion for S_0^{-1} follows from Lemma 6.6 and $S_0 S_0^{-1} = 1$. \square

From Theorem 5.1 and Lemma 6.3 one can write

$$p_\psi - \tilde{p}(z - i\epsilon H_\psi) = \sqrt{\epsilon} r + r_0$$

where $r \in \tilde{S}(\omega^{-1}\langle \xi \rangle_\gamma^\kappa h, \bar{g})$ with $|r| \leq C\epsilon\omega^{-1}\langle \xi \rangle_\gamma^\kappa h$ and $r_0 \in S(\langle \xi \rangle_\gamma^{m-\delta(m+1)}, \bar{g})$. Thus $r\# \bar{Q} \in \tilde{S}(\omega^{-1}\langle \xi \rangle_\gamma^\kappa h^2, \bar{g})$ and $|r\bar{Q}| \leq C\epsilon\omega^{-1}\langle \xi \rangle_\gamma^\kappa h^2$. On the other hand Lemma 6.1 shows

$$\langle \xi \rangle_\gamma^{m-\delta(m+1)} = \langle \xi \rangle_\gamma^{m-1+\kappa-\delta(m-1)} \leq C\epsilon^{1-m}\gamma^{-\delta}\omega^{-1}\langle \xi \rangle_\gamma^\kappa h$$

so that we see $r_0 \in S(\omega^{-1}\langle \xi \rangle_\gamma^\kappa h, \bar{g})$ and $|r_0| \leq C\epsilon^{3/2}\omega^{-1}\langle \xi \rangle_\gamma^\kappa h$ for $\gamma \geq \gamma_0(\epsilon)$. Therefore in virtue of Lemma 6.7 there is $\tilde{r} \in \tilde{S}(1, \bar{g})$ with $|\tilde{r}| \leq C\sqrt{\epsilon}$ such that

$$\text{Im } (p_\psi \# \bar{Q}) = S_0(1 - \tilde{r})$$

in some conic neighborhood of ρ .

We turn to $R = \sum_{j=0}^{m-1} (P_j)_\psi \in S(\langle \xi \rangle_\gamma^{m-1}, \bar{g})$. From Lemma 6.1 again we have

$$\begin{aligned} \langle \xi \rangle_\gamma^{m-1} &\leq C\epsilon^{1-m}\omega^{-(m-1)}h = C\epsilon^{1-m}\omega^{-1}\langle \xi \rangle_\gamma^\kappa h(\omega^{-(m-2)}\langle \xi \rangle_\gamma^{-\kappa}) \\ &\leq C\epsilon^{1-m}\gamma^{-\kappa+\delta(m-2)}\omega^{-1}\langle \xi \rangle_\gamma^\kappa h. \end{aligned}$$

Recalling that $\kappa - \delta(m-2) = \epsilon^* > 0$ and hence we can assume $C\epsilon^{1-m}\gamma^{-\epsilon^*} \leq C\epsilon^{3/2}$ for $\gamma \geq \gamma_0(\epsilon)$ so that there exists $\hat{r} \in \tilde{S}(1, \bar{g})$ with $|\hat{r}| \leq C\sqrt{\epsilon}$ such that $\text{Im}(R\#\bar{Q}) = S_0(1 - \hat{r})$ in a conic neighborhood of ρ . Thus we conclude

$$\begin{cases} \text{Im}(P_\psi\#\bar{Q}) = E^2, & E \in \tilde{S}(\omega^{-1/2}\langle \xi \rangle_\gamma^{\kappa/2}h, \bar{g}), \\ \epsilon^{1/2}\omega^{-1/2}\langle \xi \rangle_\gamma^{\kappa/2}h/C \leq |E| \leq C\epsilon^{1/2}\omega^{-1/2}\langle \xi \rangle_\gamma^{\kappa/2}h \end{cases}$$

in a conic neighborhood of ρ with C independent of ϵ . The rest of the proof of deriving microlocal energy estimates is just a repetition of the arguments in [13] and we conclude that the Cauchy problem for $p + P_{m-1} + \dots$ is $\gamma^{(1/\kappa)}$ well-posed at the origin. Note that $1/\kappa = m/(m-2+2\epsilon^*)$ and $\epsilon^* > 0$ is arbitrarily small so that $1/\kappa$ is as close to $m/(m-2)$ as we please.

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Department of Mathematics, Osaka University, Machikaneyama 1-1, Toyonaka, 560-0043, Osaka, Japan; nishitani@math.sci.osaka-u.ac.jp